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NONZERO-SUM SUBMODULAR MONOTONE-FOLLOWER GAMES: EXISTENCE AND APPROXIMATION OF NASH EQUILIBRIA

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Abstract. We consider a class of N-player stochastic games of multi-dimensional singular control, in which each player faces a minimization problem of monotone-follower type with submodular costs. We call these games *monotone-follower games*. In a not necessarily Markovian setting, we establish the existence of Nash equilibria. Moreover, we introduce a sequence of approximating games by restricting, for each $n \in \mathbb{N}$, the players' admissible strategies to the set of Lipschitz processes with Lipschitz constant bounded by n. We prove that, for each $n \in \mathbb{N}$, there exists a Nash equilibrium of the approximating game and that the sequence of Nash equilibria converges, in the Meyer-Zheng sense, to a weak (distributional) Nash equilibrium of the original game of singular control. As a byproduct, such a convergence also provides approximation results of the equilibrium values across the two classes of games. We finally show how our results can be employed to prove existence of open-loop Nash equilibria in an N-player stochastic differential game with singular controls, and we propose an algorithm to determine a Nash equilibrium for the monotone-follower game.

Keywords: nonzero-sum games; singular control; submodular games; Meyer-Zheng topology; maximum principle; Nash equilibrium; stochastic differential games; monotone-follower problem.

AMS subject classification: 91A15, 06B23, 49J45, 60G07, 91A23, 93E20.

1. Introduction

We consider a class of stochastic N-player games over a finite time-horizon in which each player, indexed by $i = 1, ..., N$, faces a multi-dimensional singular stochastic control problem of monotone-follower type. On a complete probability space, consider a multi-dimensional càdlàg (i.e., right-continuous with left limits) process L and, for $i = 1, ..., N$, multi-dimensional continuous semimartingales f^i with nonnegative components. Denote by $\mathbb{F}$ the right-continuous extension of the filtration generated by $f = (f^1, ..., f^N)$ and L, augmented by the sets of zero probability. We call *monotone-follower game* the game in which each player i is allowed to choose a multi-dimensional control A^i in the set of *admissible strategies*

$$\mathcal{A} := \{\mathbb{F}\text{-adapted processes with nondecreasing, nonnegative and càdlàg components}\},$$

in order to minimize the cost functional

$$\mathcal{J}^i(A^i, A^{-i}) := \mathbb{E}\bigg[\int_0^T h^i(L_t, A^i_t, A^{-i}_t)\, dt + g^i(L_T, A^i_T, A^{-i}_T) + \int_{[0,T]} f^i_t \, dA^i_t\bigg],$$

where $A^{-i} := (A^j)_{j\neq i}$. Here $T < \infty$ and h^i and g^i are suitable nonnegative convex cost functions.

Next, we introduce a sequence of approximating games with regular controls in the following way. For each $n \in \mathbb{N}$, define the *n-Lipschitz game* as the game in which players are restricted to pick a Lipschitz control in the set of *admissible n-Lipschitz strategies*

$$\mathcal{L}(n) = \{A \in \mathcal{A} \,|\, A \text{ is Lipschitz with Lipschitz constant smaller that } n \text{ and } A_0 = 0\},$$

in order to minimize the cost functionals $\mathcal{J}^i$.

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Our main contributions are the following.

(1) Under submodularity conditions on the functions h^i and g^i, we establish the existence of Nash equilibria for the monotone-follower and the n-Lipschitz games.

(2) We show connections across these two classes of games. In particular:
   (i) any sequence obtained by choosing, for each $n \in \mathbb{N}$, a Nash equilibrium of the n-Lipschitz game is relatively compact in the Meyer-Zheng topology, and any accumulation point of this sequence is the law of a *weak Nash equilibrium* of the monotone-follower game (see Definition 4 below). That is, any accumulation point is a Nash equilibrium on a suitable probability space on which are defined processes $\bar{f}$ and $\bar{L}$ such that their joint law coincides with the joint law of f and L;
   (ii) the N-dimensional vector whose components are the expected costs associated to any weak Nash equilibrium obtained through the previous approximation is a *Nash equilibrium payoff*. Moreover, for each $\varepsilon > 0$, there exist $n_\varepsilon \in \mathbb{N}$ large enough and a Nash equilibium of the n_ε-Lipschitz game which is an ε-Nash equilibrium of the monotone-follower game.

Furthermore: we provide applications of our results to deduce existence of Nash equilibria for a class of stochastic differential games with singular controls and non-Markovian random costs; in the spirit of [67], we construct an algorithm to determine a Nash equilibrium of the monotone-follower game; we provide an existence result for the monotone-follower game in which players are allowed to choose both a regular control and a singular control.

To the best of our knowledge, general existence and approximation results of Nash equilibria in N-player non-Markovian stochastic games of multi-dimensional singular control appear in this paper for the first time.

1.1. **Background literature.** A singular stochastic control problem appears for the first time in [10], where the problem of controlling the motion of a spaceship has been addressed. Later on, examples of solvable singular stochastic control problems have been studied in [11].

Singular stochastic control problems of monotone-follower type have been introduced and studied in [40] and [42]. A monotone-follower problem is the problem of tracking a stochastic process by a nondecreasing process in order to optimize a certain performance criterion. Since then, this class of problems has found many applications in economics and finance (see [9], [21], [23], [26], [53], among many others), operations research (see, e.g., [32] and [36]), queuing theory (see, e.g., [44]), mathematical biology (see, e.g., [2] and [3]), aerospace engineering (see, e.g., [52]), and insurance mathematics (see [49], [63], and [64], among others).

The literature on singular stochastic control problems experienced results on existence of minima (or maxima) (see [18], [29] and [37], among others), characterization of the optimizers through first order conditions (see, e.g.,[8], [9], [20] and [55]), as well as connections to optimal stopping problems (see, e.g., [42] or the more recent [15], [19], [59]) and to constrained backward stochastic differential equations [16]. We also mention the recent work [47], as their version of the monotone-follower problem is the single-agent version (in weak formulation) of our game.

The number of contributions on games of singular controls is still quite limited (see [27], [30], [34], [35], [38], [45], [46], [65], [71]), although these problems have received an increasing interest in the last years. We briefly discuss here some of these works. In [65] it is determined a symmetric Nash equilibrium of a monotone-follower game with symmetric payoffs (i.e., the cost functional is the same for all players), and it is provided a characterization of any equilibria through a system of first order conditions. The same approach is also followed in [30] for a game in which players are allowed to choose a regular control and a singular control. Such a problem has been motivated by a question arising in public economic theory.

A general characterization of Nash equilibria through the Pontryagin Maximum Principle approach has been investigated in the recent [71] for regular-singular stochastic differential games. Connections between nonzero-sum games of singular control and games of optimal stopping have been tackled in [27]. It is also worth mentioning some recent works on mean field games with singular controls (see [31] and [33]) and their connection to symmetric N-player games (see [35]). A complete analysis of a Markovian N-player stochastic game in which players can control an underlying diffusive dynamic through a control of bounded-variation is provided in the recent [34]. There, the authors derive a Nash equilibrium by solving a system of *moving* free boundary problems. General existence result for stochastic games with multi-dimensional singular controls and non-Markovian costs were, however, missing in the literature, and this has motivated our study.

1.2. **Our results.** We now provide more details on our results by discussing the ideas and techniques of their proofs.

The existence results. Going back to the seminal work of J. Nash [58], a typical way to prove existence of Nash equilibria is to show existence of a fixed point for the best reply map. In the spirit of [67], our strategy to prove existence of Nash equilibria in the monotone-follower game and in the n-Lipschitz game is to exploit the submodular structure of our games in order to apply a lattice-theoretical fixed point theorem: the Tarski's fixed point theorem (see [66]). We proceed as follows. We first endow the spaces of admissible strategies $\mathcal{A}$ and $\mathcal{L}(n)$ (defined above) with a lattice structure. While the lattice $\mathcal{L}(n)$ is complete, the same does not hold true for $\mathcal{A}$. To overcome this problem, we show that, under suitable assumptions, each "reasonable" strategy lives in a bounded subset of $\mathcal{A}$, and we restrict our analysis to this subset, which is in fact a complete lattice. We then prove that the best reply maps are non empty. To accomplish this task in the n-Lipschitz game, we employ the so-called classical *direct method*. Indeed, since each strategy is forced to be n-Lipschitz, then the sequence of time-derivatives of any minimizing sequence is bounded in $\mathbb{L}^2$. Hence, Banach-Saks' theorem, together with the lower semi-continuity and the convexity of the costs, allows to conclude existence of the minima. On the other hand, for the monotone-follower game we use some more recent techniques already employed to prove existence of optimizers in singular stochastic control problems (see [9] and [61]). Assuming a uniform coercivity condition on the costs (which is, anyway, necessary for existence of Nash equilibria; see Remark 2.6 below) we can use a theorem by Y.M. Kabanov (see Lemma 3.5 in [39]) which gives relative sequential compactness, in the Cesàro sense, of any minimizing sequence. Then, exploiting again the lower semi-continuity and the convexity of the cost functions, we conclude existence of the minima. Next, we show that the best reply maps preserve the order in the spaces of admissible strategies, and for this the submodular condition is essential. The existence result then follows by invoking Tarski's fixed point theorem.

Our finding also generalizes also to the infinite time-horizon case and to the monotone-follower game in which players are allowed to choose both a regular control and a singular control. Moreover, some of the assumptions can be removed if we impose *finite fuel constraints.*

It is worth stressing that our proof strongly hinges on the submodularity assumption, which is, however, a typical requirement in many problems arising in applications (see, e.g., [56], [57], [67], [69], the more recent [6] and [7], or the books [68] and [70] and the references therein).

The approximation results. Singular control problems naturally arise to overcome the ill-posedness of standard stochastic control problems in which the control affects linearly the dynamics of the state variable, and the cost of control is proportional to the effort. Some kind of connection between regular control problems with the linear structure described above and singular control problems is then expected, and actually already discussed in the literature (see, e.g, the early [51] and [52] for an analytical approach, and [47] for a probabilistic approach). In Theorem 21 of [47], it is shown that any sequence obtained by choosing, for each $n \in \mathbb{N}$, a minimizer of the monotone-follower problem when the class of admissible controls is restricted to the set of n-Lipschitz controls, suitably approximates a (weak) optimal solution to the original monotone-follower problem.

In our game-setting, we prove that any sequence of Nash equilibria of the n-Lipschitz game is weakly relatively compact, and that any accumulation point is a weak Nash equilibrium of the monotone-follower game. We first show that this sequence satisfies a tightness criterion for the Meyer-Zheng topology. Then, we prove that any Nash equilibrium of the n-Lipschitz game necessarily satisfies a system of stochastic equations. After changing the underlying probability space by a Skorokhod representation, we pass to the limit in these systems of equations and we deduce that any accumulation point solves a new system of stochastic equations. These equations can be viewed as a version of the Pontryagin maximum principle, and they are sufficient to ensure that the limit point is a Nash equilibrium in the new probability space, hence a weak Nash equilibrium.

As a byproduct of this result, we are able to show that, for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ large enough such that the Nash equilibrium of the n-Lipschitz game is an ε-Nash equilibrium of the monotone-follower game. This gives a clearer interpretation of the weak Nash equilibrium found through the approximation: the N-dimensional vector whose components are the expected costs associated to the weak Nash equilibrium is, in fact, a *Nash equilibrium payoff* (as defined in [17]) of the monotone-follower game.

Applications and examples. Our existence result applies to deduce existence of open-loop Nash equilibria in stochastic differential games with singular controls and non-Markovian random costs, whenever a certain structure is preserved by the dynamics. For the sake of illustration, we consider the case in which the dynamics of the state variable of each player are a linearly controlled geometric Brownian motion and a linearly controlled Ornstein–Uhlenbeck process.

Moreover, we consider the algorithm introduced by Topkis (see Algorithm II in [67]) for submodular games: given as initial point the constantly null profile strategy, this algorithm consists of an iteration of the best reply map. We show that, also in our setting with singular controls, this algorithm converges to a Nash equilibrium.

1.3. **Organization of the paper.** In Section 2.1 we introduce the monotone-follower game. Sections 2.2 and 3 are devoted to the existence theorems of Nash equilibria for the submodular monotone-follower game and for the n-Lipschitz game, respectively. The approximation results are contained in Section 4. The application of our result to suitable stochastic differential games is provided in Section 5, together with the proof of the convergence to a Nash equilibrium of a certain algorithm. Section 6 contains an extension of the existence result to games with both regular and singular controls. In Appendix A we collect some technical lemmata and some proofs of results from Section 4, while Appendix B is devoted to recall some results about the Meyer-Zheng topology.

2. The Monotone-Follower Game

2.1. **Definition of the Monotone-Follower Game.** Fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a finite time horizon $T \in (0, \infty)$, an integer $N \geq 2$ and $k, d \in \mathbb{N}$. Consider a stochastic process $L : \Omega \times [0, T] \to \mathbb{R}^k$, and, for $i = 1, ..., N$, assume to be given continuous semimartingales $f^i : \Omega \times [0, T] \to \mathbb{R}^d_+$, and set $f := (f^1, ..., f^N)$. Denote by $\bar{\mathbb{F}}^{f,L}_+ = \{\bar{\mathcal{F}}^{f,L}_{t+}\}_{t\in[0,T]}$ the right-continuous extension of the filtration generated by f and L, augmented by the $\mathbb{P}$-null sets.

Define the space of *admissible strategies*

$$\mathcal{A} := \left\{ V : \Omega \times [0,T] \to \mathbb{R}^d \;\middle|\; \begin{array}{c} V \text{ is an } \bar{\mathbb{F}}_+^{f,L}\text{-adapted càdlàg process, with} \\ \text{nondecreasing and nonnegative components} \end{array} \right\}, \tag{2.1}$$

and let $\mathcal{A}^N := \bigotimes_{i=1}^N \mathcal{A}$ denote the set of *admissible profile strategies.* In order to avoid confusion, in the following we will denote profile strategies in bold letters.

For each $i = 1, ..., N$, consider measurable functions $h^i, g^i : \mathbb{R}^k \times \mathbb{R}^{Nd} \to [0, \infty)$. We define the *monotone-follower game* as the game in which each player $i \in \{1, ..., N\}$ is allowed to choose an admissible strategy $A^i \in \mathcal{A}$ in order to minimize the cost functional

$$\mathcal{J}^i(A^i, A^{-i}) := \mathbb{E}[C^i(f, L, \mathbf{A})] := \mathbb{E}\bigg[\int_0^T h^i(L_t, \mathbf{A}_t)\, dt + g^i(L_T, \mathbf{A}_T) + \int_{[0,T]} f_t^i \, dA_t^i \bigg], \tag{2.2}$$

where $A^{-i} := (A^j)_{j \neq i}$ and $\mathbf{A} := (A^i, A^{-i}) \in \mathcal{A}^N$. Here and in the sequel the integrals with respect to A^i are defined by

$$\int_{[0,T]} f_t^i \, dA_t^i := f_0^i A_0^i + \int_0^T f_t^i \, dA_t^i = \sum_{\ell=1}^d f_0^{\ell,i} A_0^{\ell,i} + \sum_{\ell=1}^d \int_0^T f_t^\ell \, dA_t^{\ell,i},$$

where the integrals on the right hand side are intended in the standard Lebesgue-Stieltjes sense on the interval $(0, T]$.

We recall the notion of Nash equilibrium.

Definition 1. *An admissible profile strategy $\bar{\mathbf{A}} \in \mathcal{A}^N$ is a Nash equilibrium if, for every $i = 1, ..., N$, we have $\mathcal{J}^i(\bar{\mathbf{A}}) < \infty$ and*

$$\mathcal{J}^i(\bar{A}^i, \bar{A}^{-i}) \leq \mathcal{J}^i(V^i, \bar{A}^{-i}), \quad \text{for every} \quad V^i \in \mathcal{A}.$$

Letting $2^{\mathcal{A}}$ denote the set of all subset of $\mathcal{A}$, for each $i = 1, ..., N$ define the *best reply map* $R^i : \mathcal{A}^N \to 2^{\mathcal{A}}$ by

$$R^i(\mathbf{A}) := \underset{V^i \in \mathcal{A}}{\arg\min}\, \mathcal{J}^i(V^i, A^{-i}). \tag{2.3}$$

Moreover define the map

$$\mathbf{R} := (R^1, ..., R^N) : \mathcal{A}^N \to \bigotimes_{i=1}^N 2^{\mathcal{A}}, \tag{2.4}$$

and notice that the set of Nash equilibria coincides with the set of fixed points of the map $\mathbf{R}$; that is, the set of $\bar{\mathbf{A}} \in \mathcal{A}^N$ such that $\bar{\mathbf{A}} \in \mathbf{R}(\bar{\mathbf{A}})$.

Remark 2.1. *The notion of equilibrium introduced above is that of the so-called Open-Loop Nash equilibrium. We focus on this specific class of equilibria since serious conceptual –so far unsolved – problems arise when one tries to define a game of singular controls with Closed-Loop strategies (see* [5] *for a discussion, and also* [30] *and* [65]*).*

In the rest of this paper, for $m \in \mathbb{N}$ and $x, y \in \mathbb{R}^m$, we denote by xy the scalar product in $\mathbb{R}^m$, as well as by $|\cdot|$ the Euclidean norm in $\mathbb{R}^m$. For $x, y \in \mathbb{R}^m$ and $c \in \mathbb{R}$, we will write $x \leq y$ if $x^\ell \leq y^\ell$ for each $\ell = 1, ..., m$, as well as $x \leq c$ if $x^\ell \leq c$ for each $\ell = 1, ..., m$. Moreover, we set $x \wedge y := (x^1 \wedge y^1, ..., x^m \wedge y^m)$ and $x \vee y := (x^1 \vee y^1, ..., x^m \vee y^m)$, where $x^\ell \wedge y^\ell := \min\{x^\ell, y^\ell\}$ and $x^\ell \vee y^\ell := \max\{x^\ell, y^\ell\}$ for each $\ell = 1, ..., m$.

We now specify the structural hypothesis on the costs.

Assumption 2.2. *For each $i = 1, ..., N$, assume that:*

1. *for each $(l, a^{-i}) \in \mathbb{R}^k \times \mathbb{R}^{(N-1)d}$, the functions $h^i(l, \cdot, a^{-i})$ and $g^i(l, \cdot, a^{-i})$ are lower semi-continuous, and strictly convex;*

(2) *for each* $l \in \mathbb{R}^k$ *the functions* $h^i(l,\cdot,\cdot)$ *and* $g^i(l,\cdot,\cdot)$ *have decreasing differences in* (a^i, a^{-i}), *i.e.*

$$h^i(l,\bar{a}^i,a^{-i}) - h^i(l,a^i,a^{-i}) \geq h^i(l,\bar{a}^i,\bar{a}^{-i}) - h^i(l,a^i,\bar{a}^{-i}),$$
$$g^i(l,\bar{a}^i,a^{-i}) - g^i(l,a^i,a^{-i}) \geq g^i(l,\bar{a}^i,\bar{a}^{-i}) - g^i(l,a^i,\bar{a}^{-i}),$$

for each $a, \bar{a} \in \mathbb{R}^{Nd}$ *such that* $\bar{a} \geq a$;

(3) *for each* $(l, a^{-i}) \in \mathbb{R}^k \times \mathbb{R}^{(N-1)d}$, *the functions* $h^i(l,\cdot,a^{-i})$ *and* $g^i(l,\cdot,a^{-i})$ *are submodular, i.e.*

$$h^i(l,\bar{a}^i,a^{-i}) + h^i(l,a^i,a^{-i}) \geq h^i(l,\bar{a}^i \wedge a^i,a^{-i}) + h^i(l,\bar{a}^i \vee a^i,a^{-i}),$$
$$g^i(l,\bar{a}^i,a^{-i}) + g^i(l,a^i,a^{-i}) \geq g^i(l,\bar{a}^i \wedge a^i,a^{-i}) + g^i(l,\bar{a}^i \vee a^i,a^{-i}),$$

for each $a, \bar{a} \in \mathbb{R}^{Nd}$.

Under Conditions 2 and 3 of Assumption 2.2 we refer to the game introduced above as to the *submodular monotone-follower game* (see [67] for a static deterministic N-player game submodular game). The submodular structure of our game will play a fundamental role in the following.

Remark 2.3. *Condition 3 of Assumption 2.2 is verified if and only if, for each* $(l, a^{-i}) \in \mathbb{R}^k \times \mathbb{R}^{(N-1)d}$ *and* $\ell = 1, ..., d$, $h^i(l,\cdot,a^{-i})$ *and* $g^i(l,\cdot,a^{-i})$ *have decreasing differences in* $(a^{\ell,i}, a^{-\ell,i})$, *where* $a^{-\ell,i} = (a^{j,i})_{j\neq \ell}$ *(see Theorem 2.6.1 and Corollary 2.6.1 at p. 44 in [68]). Hence, in the case of twice-differentiable functions, this condition corresponds to the nonpositivity of the second order mixed derivatives; that is*

$$\frac{\partial^2 h^i}{\partial a^{\ell,i}\partial a^{j,i}} \leq 0 \quad \text{and} \quad \frac{\partial^2 g^i}{\partial a^{\ell,i}\partial a^{j,i}} \leq 0 \quad \text{for each} \quad i = 1, ..., N \quad \text{and} \quad \ell \neq j.$$

2.2. **Existence of Nash Equilibria in the Submodular Monotone-Follower Game.** On the space of admissible strategies $\mathcal{A}$ (cf. Definition 2.1) we define the order relation $\preccurlyeq$ such that, for $V, U \in \mathcal{A}$, one has

$$V \preccurlyeq U \quad \iff \quad V_t \leq U_t \quad \forall\, t \in [0,T], \quad \mathbb{P}-\text{a.s.}$$

Moreover, we can endow the space $\mathcal{A}$ with a lattice structure, defining the processes $V \wedge U$ and $V \vee U$ as

$$(V \wedge U)_t := V_t \wedge U_t \quad \text{and} \quad (V \vee U)_t := V_t \vee U_t \quad \forall t \in [0,T], \quad \mathbb{P}-\text{a.s.}$$

In the same way, on the set of profile strategies $\mathcal{A}^N$, define, for $\mathbf{A}, \mathbf{B} \in \mathcal{A}^N$, an order relation $\preccurlyeq^N$ by

$$\mathbf{A} \preccurlyeq^N \mathbf{B} \quad \iff \quad A^i \preccurlyeq B^i \quad \forall\, i \in \{1, ..., N\},$$

together with the lattice structure

$$\mathbf{A} \wedge \mathbf{B} := (A^1 \wedge B^1, ..., A^N \wedge B^N) \quad \text{and} \quad \mathbf{A} \vee \mathbf{B} := (A^1 \vee B^1, ..., A^N \vee B^N).$$

We now provide an existence result for the submodular monotone-follower game.

Theorem 2.4. *Let Assumption 2.2 hold and assume that the following uniform coercivity condition is satisfied: there exist two constants* $K, \kappa > 0$ *such that, for each* $i = 1, ..., N$,

$$\mathcal{J}^i(A^i, A^{-i}) \geq \kappa\, \mathbb{E}[|A^i_T|] \quad \text{for all} \quad \mathbf{A} \in \mathcal{A}^N \quad \text{with} \quad \mathbb{E}[|A^i_T|] \geq K. \tag{2.5}$$

Suppose, moreover, that there exists a constant $M > 0$ *such that, for each* $i = 1, ..., N$,

$$\text{for all} \quad \mathbf{A} \in \mathcal{A}^N \quad \text{there exists} \quad r^i(\mathbf{A}) \in \mathcal{A} \quad \text{such that} \quad \mathcal{J}^i(r^i(\mathbf{A}), A^{-i}) \leq M. \tag{2.6}$$

Then the set of Nash equilibria $F \subset \mathcal{A}^N$ *is non empty, and the partially ordered set* $(F, \preccurlyeq^N)$ *is a complete lattice.*

Proof. Our aim is to prove existence of a Nash equilibrium by applying Tarski's fixed point theorem (see [66], Theorem 1) to the map $\mathbf{R}$ (cf. (2.4)). For this, the assumption on the submodularity of h^i and g^i will play a crucial role.

First of all, recalling k, K and M from (2.5) and (2.6), define the constant $w := \frac{2M}{\kappa} \vee K$, and introduce the set of restricted admissible strategies

$$\mathcal{A}(w) := \{ A \in \mathcal{A} \,|\, \mathbb{E}[A^l_T] \le w, \, \forall \, l = 1, ..., d \}, \tag{2.7}$$

and the set of restricted profile strategies as $\mathcal{A}(w)^N := \bigotimes_{i=1}^N \mathcal{A}(w)$. In the following steps we will identify the proper framework allowing us to apply Tarski's fixed point theorem.

(Step 1) The lattices $(\mathcal{A}(w)^N, \preccurlyeq^N)$ and $(\mathcal{A}(w), \preccurlyeq)$ are complete.

We prove the claim only for the lattice $(\mathcal{A}(w)^N, \preccurlyeq^N)$, since an analogous rationale applies to show that the lattice $(\mathcal{A}(w), \preccurlyeq)$ is complete.

To prove that the lattice $(\mathcal{A}(w)^N, \preccurlyeq^N)$ is complete we have to show that each subset of $\mathcal{A}(w)^N$ has a least upper bound and a greatest lower bound. We now prove only the existence of a least upper bound, since the existence of a greatest lower bound follows by similar arguments.

Consider a subset $\{\mathbf{A}^j\}_{j\in\mathcal{I}}$ of $\mathcal{A}(w)^N$, where $\mathcal{I}$ is a set of indexes. We want to show that there exists an element $\mathbf{S}$ of $\mathcal{A}(w)^N$ such that $\mathbf{A}^j \preccurlyeq^N \mathbf{S}$ for each $j \in \mathcal{I}$ and such that, if $\mathbf{B}$ is another element of $\mathcal{A}(w)^N$ with $\mathbf{A}^j \preccurlyeq^N \mathbf{B}$ for each $j \in \mathcal{I}$, then $\mathbf{S} \preccurlyeq^N \mathbf{B}$.

Define $Q := ([0,T] \cap \mathbb{Q}) \cup \{T\}$. For each $q \in Q$ we set

$$\tilde{\mathbf{S}}_q := \operatorname*{ess\,sup}_{j\in\mathcal{I}} \mathbf{A}^j_q, \tag{2.8}$$

and we recall that there exists a countable subset $\mathcal{I}_q$ of $\mathcal{I}$ such that

$$\tilde{\mathbf{S}}_q = \sup_{j\in\mathcal{I}_q} \mathbf{A}^j_q. \tag{2.9}$$

Define next the right-continuous process $\mathbf{S} : \Omega \times [0,T] \to [0,\infty)^{Nd}$ by

$$\mathbf{S}_T := \tilde{\mathbf{S}}_T, \quad \text{and} \quad \mathbf{S}_t := \inf\{ \tilde{\mathbf{S}}_q \,|\, q > t, q \in Q\}, \quad \text{for} \quad t < T. \tag{2.10}$$

Observe that, being $\mathbf{S}_T = \sup_{j\in\mathcal{I}_T} \mathbf{A}^j_T$ $\mathbb{P}$-a.s., by Fatou's lemma it follows that $\mathbb{E}[\mathbf{S}^{\ell,i}_T] \le w$ for each $l = 1, ..., d$ and $i = 1, ..., N$; that is, $\mathbf{S} \in \mathcal{A}^N$. Moreover, $\mathbf{S}$ is adapted to $\bar{\mathbb{F}}^{f,L}_+$. Indeed, by its definition, $\mathbf{S}_T$ is clearly $\bar{\mathcal{F}}^{f,L}_T$-measurable. On the other hand, if $t < T$, for a generic $\bar{q} \in Q$ with $\bar{q} > t$, we have that

$$\mathbf{S}_t = \inf\{ \tilde{\mathbf{S}}_q \,|\, q > t, \, q \in Q\} = \inf\{ \tilde{\mathbf{S}}_q \,|\, t < q \le \bar{q}, \, q \in Q\},$$

where we have used that the process $\{\tilde{\mathbf{S}}_q\}_{q\in Q}$ is increasing. Since the right-hand side of the latter equation is $\bar{\mathcal{F}}^{f,L}_{\bar{q}}$-measurable, we deduce that $\mathbf{S}_t$ is $\bar{\mathcal{F}}^{f,L}_{\bar{q}}$-measurable for each $\bar{q} > t$, and this implies, by the right-continuity of $\bar{\mathbb{F}}^{f,L}_+$, that $\mathbf{S}_t$ is $\bar{\mathcal{F}}^{f,L}_t$-measurable. Finally, since $\mathbf{S}$ is clearly increasing, nonnegative and right-continuous, we conclude that $\mathbf{S} \in \mathcal{A}(w)^N$.

Take now $j \in \mathcal{I}$. From the definition (2.8) of $\tilde{\mathbf{S}}$, for each $q \in Q$ we have $\mathbf{A}^j_q \le \tilde{\mathbf{S}}_q$ $\mathbb{P}$-a.s., which means that there exists a $\mathbb{P}$-null set $\mathcal{N}_q$ such that $\mathbf{A}^j_q(\omega) \le \tilde{\mathbf{S}}_q(\omega)$ for each $\omega \in \Omega \setminus \mathcal{N}_q$. Defining then the $\mathbb{P}$-null set $\mathcal{N} := \bigcup_{q\in Q} \mathcal{N}_q$, we have $\mathbf{A}^j_q(\omega) \le \tilde{\mathbf{S}}_q(\omega)$ for each $\omega \in \Omega \setminus \mathcal{N}$ and $q \in Q$, which, by right-continuity, in turn implies that $\mathbf{A}^j_t(\omega) \le \mathbf{S}_t(\omega)$ for each $\omega \in \Omega \setminus \mathcal{N}$ and $t \in [0,T]$. Hence, $\mathbf{A}^j \preccurlyeq^N \mathbf{S}$ for each $j \in \mathcal{I}$ as desired.

Consider next an element $\mathbf{B}$ of $\mathcal{A}(w)^N$ such that $\mathbf{A}^j \preccurlyeq^N \mathbf{B}$ for each $j \in \mathcal{I}$. For $q \in Q$ and $j \in \mathcal{I}_q$ there exists a $\mathbb{P}$-null set $\mathcal{M}^j_q$ such that $\mathbf{A}^j_q(\omega) \le \mathbf{B}_q(\omega)$ for all $\omega \in \Omega \setminus \mathcal{M}^j_q$. Defining then $\mathcal{M}_q := \bigcup_{j\in\mathcal{I}_q} \mathcal{M}^j_q$, we have $\mathbf{A}^j_q(\omega) \le \mathbf{B}_q(\omega)$ for all $\omega \in \Omega \setminus \mathcal{M}_q$ and $j \in \mathcal{I}_q$, which, by (2.9), implies that $\tilde{\mathbf{S}}_q(\omega) \leq \mathbf{B}_q(\omega)$ for all $\omega \in \Omega \setminus \mathcal{M}_q$. Finally, introducing the $\mathbb{P}$-null set $\mathcal{M} := \bigcup_{q\in Q} \mathcal{M}_q$, we have $\tilde{\mathbf{S}}_q(\omega) \leq \mathbf{B}_q(\omega)$ for all $\omega \in \Omega \setminus \mathcal{M}$ and $q \in Q$, and, by right-continuity, we deduce that $\mathbf{S} \preccurlyeq^N \mathbf{B}$.

(Step 2) The best reply maps $R^i : \mathcal{A}^N \to \mathcal{A}(w)$ are well defined.
Fix i and take $\mathbf{A} \in \mathcal{A}^N$. We have to prove that there exists a unique $B \in \mathcal{A}$ such that

$$\mathcal{J}^i(B, A^{-i}) = \min_{V\in\mathcal{A}} \mathcal{J}^i(V, A^{-i}),$$

and, moreover, that $B \in \mathcal{A}(w)$. Clearly, by (2.3), we have $B = \{R^i(\mathbf{A})_t\}_{t\in[0,T]}$.

Let $\{V^j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence for the functional $\mathcal{J}^i(\cdot, A^{-i})$. Fix $\delta > 0$ and, for each $j \in \mathbb{N}$, let $\tilde{V}^j$ denote

$$\tilde{V}^j_t := \begin{cases} 0 & \text{if } t \in [-\delta, 0) \\ V^j_t & \text{if } t \in [0, T]. \end{cases}$$

Thanks to the coercivity conditions (2.5) on the costs, we deduce that

$$\sup_{j\in\mathbb{N}} \mathbb{E}[|\tilde{V}^j_T|] = \sup_{j\in\mathbb{N}} \mathbb{E}[|V^j_T|] < \infty.$$

We can then use (a minimal adjustment to $[-\delta, T]$ of) Lemma 3.5 in [39], to find a càdlàg, nondecreasing, nonnegative, $\bar{\mathbb{F}}^{f,L}_+$-adapted process B, and a subsequence of $\{\tilde{V}^j\}_{j\in\mathbb{N}}$ (not relabeled) such that, $\mathbb{P}$-a.s.,

$$\lim_m \int_{-\delta}^T \varphi_t \, dB^m_t = \int_{-\delta}^T \varphi_t \, dB_t \quad \forall \varphi \in \mathcal{C}_b((-\delta, T); \mathbb{R}^d) \quad \text{and} \quad \lim_m B^m_T = B_T, \tag{2.11}$$

where we set, $\mathbb{P}$-a.s.

$$B^m_t := \frac{1}{m} \sum_{j=1}^m \tilde{V}^j_t, \qquad \forall t \in [-\delta, T]. \tag{2.12}$$

For each test functions $\varphi \in \mathcal{C}_b((-\delta, T); \mathbb{R}^d)$ with compact support contained in $(-\delta, 0)$, defining $\Phi_t := \int_0^t \varphi_s \, ds$, we find, $\mathbb{P}$-a.s., $\int_{-\delta}^T \Phi_t \, dB_t = -\int_{-\delta}^T \varphi_t B_t \, dt = 0$. This implies, by the fundamental lemma of Calculus of Variations (see Theorem 1.24 at p. 26 in [25]), that $\mathbb{P}[B_t = 0, \forall t \in (-\delta, 0)] = 1$ and hence, by right-continuity, that $\mathbb{P}[B_t = 0, \forall t \in [-\delta, 0)] = 1$. Hence, we can write

$$\begin{aligned} \mathcal{J}^i(B, A^{-i}) =& \mathbb{E}\bigg[\int_0^T h^i(L_t, B_t, A^{-i}_t) \, dt + g^i(L_T, B_T, A^{-i}_T) + \int_{[0,T]} f^i_t \, dB_t \bigg] \\ =& \mathbb{E}\bigg[\int_0^T h^i(L_t, B_t, A^{-i}_t) \, dt + g^i(L_T, B_T, A^{-i}_T) + \int_{-\delta}^T f^i_t \, dB_t \bigg]. \end{aligned} \tag{2.13}$$

Moreover, from the limit in (2.11) we have that there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that, for each $\omega \in \Omega \setminus \mathcal{N}$ there exists a subset $\mathcal{I}(\omega) \subset [-\delta, T)$ of null Lebesgue measure, such that

$$\lim_m B^m_t(\omega) = B_t(\omega) \quad \text{for each} \quad \omega \in \Omega \setminus \mathcal{N} \quad \text{and} \quad t \in [-\delta, T] \setminus \mathcal{I}(\omega).$$

The latter convergence, allows us to invoke Fatou's lemma and to deduce that

$$\mathcal{J}^i(B, A^{-i}) \leq \liminf_m \mathbb{E}\left[\int_0^T h^i(L_t, B^m_t, A^{-i}_t) \, dt + g^i(L_T, B^m_T, A^{-i}_T) + \int_{-\delta}^T f^i_t \, dB^m_t \right],$$

upon using the lower semi-continuity of the costs and equation (2.13). Finally, thanks to the convexity of h^i and g^i and to the minimizing property of V^j, from the last inequality we can conclude that

$$\begin{aligned}
\mathcal{J}^i(B, A^{-i}) &\leq \liminf_m \frac{1}{m}\sum_{j=1}^m \mathbb{E}\left[\int_0^T h^i(L_t, \tilde{V}_t^j, A_t^{-i})\,dt + g^i(L_T, \tilde{V}_T^j, A_T^{-i}) + \int_{-\delta}^T f_t^i\, d\tilde{V}_t^j\right] \\
&= \liminf_m \frac{1}{m}\sum_{j=1}^m \mathbb{E}\bigg[\int_0^T h^i(L_t, V_t^j, A_t^{-i})\,dt + g^i(L_T, V_T^j, A_T^{-i}) + \int_{[0,T]} f_t^i\, dV_t^j\bigg] \\
&= \liminf_m \frac{1}{m}\sum_{j=1}^m \mathcal{J}^i(V^j, A^{-i}) \\
&= \min_{V\in\mathcal{A}} \mathcal{J}^i(V, A^{-i}).
\end{aligned}$$

The latter yields that B is a minimizer for $\mathcal{J}^i(\cdot, A^{-i})$. In fact, B is the unique minimizer of $\mathcal{J}^i(\cdot, A^{-i})$ by strict convexity of the costs.

It remains to prove that $B \in \mathcal{A}(w)$, and to accomplish that we argue by contradiction. If there exists $l \in \{1, ..., d\}$ such that $\mathbb{E}[B_T^l] \geq w = \frac{2M}{\kappa} \vee K$, then we have $\mathbb{E}[|B_T|] \geq \frac{2M}{\kappa} \vee K$ and hence, by the coercivity condition (2.5) together with (2.6), we deduce that

$$\mathcal{J}^i(B, A^{-i}) \geq \kappa\, \mathbb{E}[|B_T|] \geq 2M > \mathcal{J}^i(r^i(\mathbf{A}), A^{-i}),$$

which contradicts the optimality of B.

(Step 3) The best reply maps R^i are increasing, i.e. if $\mathbf{A}, \bar{\mathbf{A}} \in \mathcal{A}^N$ are such that $\mathbf{A} \preccurlyeq^N \bar{\mathbf{A}}$, then $R^i(\mathbf{A}) \preccurlyeq R^i(\bar{\mathbf{A}})$.

First of all, observe that, by an integration by parts (see, e.g., Corollary 2 at p. 68 in [60]), the cost functional rewrites as

$$\mathcal{J}^i(A^i, A^{-i}) = \mathbb{E}\bigg[\int_0^T h^i(L_t, \mathbf{A}_t)\,dt + g^i(L_T, \mathbf{A}_T) - \int_0^T A_{t-}^i\, df_t^i + f_T^i A_T^i\bigg], \tag{2.14}$$

where A_{t-}^i denotes the left-limit of A_t^i. Thanks to the optimality of $R^i(\mathbf{A})$ we have the inequality

$$\mathcal{J}^i(R^i(\bar{\mathbf{A}}) \wedge R^i(\mathbf{A}), A^{-i}) - \mathcal{J}^i(R^i(\mathbf{A}), \mathbf{A}^{-i}) \geq 0, \tag{2.15}$$

which by (2.14) and setting $R^i := R^i(\mathbf{A})$ and $\bar{R}^i := R^i(\bar{\mathbf{A}})$, can be rewritten as

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \big(h^i(L_t, R_t^i \wedge \bar{R}_t^i, A_t^{-i}) - h^i(L_t, R_t^i, A_t^{-i})\big)\,dt\right] \\
&\quad + \mathbb{E}\left[g^i(L_T, R_T^i \wedge \bar{R}_T^i, A_T^{-i}) - g^i(L_T, R_T^i, A_T^{-i})\right] \\
&\quad - \mathbb{E}\left[\int_0^T (R_{t-}^i \wedge \bar{R}_{t-}^i - R_{t-}^i)\,df_t^i\right] + \mathbb{E}\left[f_T^i(R_T^i \wedge \bar{R}_T^i - R_T^i)\right] \geq 0,
\end{aligned}$$

By the submodularity Condition 3 in Assumption 2.2, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \big(h^i(L_t, R_t^i \wedge \bar{R}_t^i, A_t^{-i}) - h^i(L_t, R_t^i, A_t^{-i})\big)\,dt\right] \\
&\qquad \leq \mathbb{E}\left[\int_0^T \big(h^i(L_t, \bar{R}_t^i, A_t^{-i}) - h^i(L_t, R_t^i \vee \bar{R}_t^i, A_t^{-i})\big)\,dt\right],
\end{aligned} \tag{2.16}$$

and

$$
\begin{aligned}
(2.17) \qquad & \mathbb{E}\left[g^i(L_T, R^i_T \wedge \bar{R}^i_T, A^{-i}_T) - g^i(L_T, R^i_T, A^{-i}_T)\right] \\
& \leq \mathbb{E}\left[g^i(L_T, \bar{R}^i_T, A^{-i}_T) - g^i(L_T, R^i_T \vee \bar{R}^i_T, A^{-i}_T)\right].
\end{aligned}
$$

Moreover, one can easily verify that

$$
\mathbb{E}\left[\int_0^T (R^i_{t-} \wedge \bar{R}^i_{t-} - R^i_{t-})\, df^i_t\right] = \mathbb{E}\left[\int_0^T (\bar{R}^i_{t-} - R^i_{t-} \vee \bar{R}^i_{t-})\, df^i_t\right] \tag{2.18}
$$

and

$$
\mathbb{E}\left[f^i_T(R^i_T \wedge \bar{R}^i_T - R^i_T)\right] = \mathbb{E}\left[f^i_T(\bar{R}^i_T - R^i_T \vee \bar{R}^i_T)\right]. \tag{2.19}
$$

Using (2.16)-(2.19) we obtain

$$
\mathcal{J}^i(R^i(\bar{\mathbf{A}}) \wedge R^i(\mathbf{A}), A^{-i}) - \mathcal{J}^i(R^i(\mathbf{A}), A^{-i}) \leq \mathcal{J}^i(R^i(\bar{\mathbf{A}}), A^{-i}) - \mathcal{J}^i(R^i(\mathbf{A}) \vee R^i(\bar{\mathbf{A}}), A^{-i}),
$$

so that, by (2.15), we deduce that

$$
\mathcal{J}^i(R^i(\bar{\mathbf{A}}), A^{-i}) - \mathcal{J}^i(R^i(\mathbf{A}) \vee R^i(\bar{\mathbf{A}}), A^{-i}) \geq 0. \tag{2.20}
$$

Now, by Condition 2 in Assumption 2.2, we have

$$
\mathcal{J}^i(R^i(\bar{\mathbf{A}}), \bar{A}^{-i}) - \mathcal{J}^i(R^i(\mathbf{A}) \vee R^i(\bar{\mathbf{A}}), \bar{A}^{-i}) \geq \mathcal{J}^i(R^i(\bar{\mathbf{A}}), A^{-i}) - \mathcal{J}^i(R^i(\mathbf{A}) \vee R^i(\bar{\mathbf{A}}), A^{-i}),
$$

and finally, by (2.20), we conclude that

$$
\mathcal{J}^i(R^i(\bar{\mathbf{A}}), \bar{A}^{-i}) - \mathcal{J}^i(R^i(\mathbf{A}) \vee R^i(\bar{\mathbf{A}}), \bar{A}^{-i}) \geq 0.
$$

Hence $R^i(\mathbf{A}) \vee R^i(\bar{\mathbf{A}})$ minimizes $\mathcal{J}^i(\cdot, \bar{A}^{-i})$ as well as $R^i(\bar{\mathbf{A}})$ and, by uniqueness, it must be $R^i(\mathbf{A}) \vee R^i(\bar{\mathbf{A}}) = R^i(\bar{\mathbf{A}})$. That is $R^i(\bar{\mathbf{A}}) \preccurlyeq R^i(\mathbf{A})$, which shows the claimed monotonicity.

(Step 5) Existence of Nash equilibria.

By the previous steps the lattice $(\mathcal{A}(w)^N, \preccurlyeq^N)$ is complete and the restriction of the map $\mathbf{R}$ (cf. (2.4)) to the set of restricted profile strategies $\mathcal{A}(w)^N$ into itself is monotone increasing. Then, by Tarski's fixed point theorem (see [66], Theorem 1), the set of fixed point of the map $\mathbf{R}$ is a non empty complete lattice. Since such a set coincides with the set of Nash equilibria, the proof is completed. □

2.3. **Some Remarks.** In this subsection we collect some remarks concerning assumptions and extensions of the previous theorem.

Remark 2.5 (Comments on the Conditions of Theorem 2.4)**.** *A few comments are worth being done.*

1. *Condition (2.5) is satisfied if, for example, there exists a constant $c > 0$ such that*

$$
\mathbb{P}\left[f^i_t \geq c,\ \forall i = 1, ..., N,\ \forall\, t \in [0,T]\right] = 1,
$$

   or if g^i are such that $g^i(l, a^i, a^{-i}) \geq \kappa\, |a^i|$.

2. *The role of Condition (2.6) is to force Nash equilibria, whenever they exist, to live in the bounded subset $\mathcal{A}^N(w)$ of $\mathcal{A}^N$. If there exist measurable functions $H, G : \mathbb{R}^k \to [0,\infty)$ such that, for each $i = 1, ..., N$ and for each $(l, a^{-i}) \in \mathbb{R}^k \times \mathbb{R}^{(N-1)d}$, we have $h^i(l, 0, a^{-i}) \leq H(l)$ and $g^i(l, 0, a^{-i}) \leq G(l)$, with*

$$
\mathbb{E}\left[\int_0^T H(L_s)\, ds + G(L_T)\right] < \infty,
$$

   then Condition (2.6) is satisfies with $r^i(\mathbf{A}) = 0$.

Remark 2.6. *Consider the case $N = 2$, $d = 1$. The costs relative to Player 1 are $f^1 = h^1 = 0$, $g^1(l, a^1, a^2) = e^{-a^1}(2 - e^{-a^2})$, while the costs of Player 2 can be generic functions satisfying our requirements. Then, all the assumptions of Theorem 2.4 are satisfied, with the exception of the coercivity condition (2.5), which is not satisfied by $\mathcal{J}^1$. If now $(\hat{A}^1, \hat{A}^2)$ were a Nash equilibrium, then for the first player we could write*

$$0 < \mathbb{E}[e^{-\hat{A}^1_T}(2 - e^{-\hat{A}^2_T})] \leq \inf_{n \in \mathbb{N}} \mathbb{E}[e^{-n}(2 - e^{-\hat{A}^2_T})] = 0,$$

which is clearly a contradiction. This example shows that, at least in the Nash equilibria, the coercivity condition (2.5) is necessarily satisfied.

Remark 2.7 (Finite-Fuel Constraint)**.** *Many models in the literature on monotone-follower problems enjoy a so-called* finite fuel constraint *(see e.g. [41] for a seminal paper, and the more recent [8] and [24]). This can be realized by requiring that the admissible control strategies stay bounded either $\mathbb{P}$-a.s. or in expectation. In our game, if we suppose that, for each $i = 1, ..., N$, the strategies of player i belongs to the set*

$$\mathcal{A}(w^i) := \{ A \in \mathcal{A} \,|\, \mathbb{E}[A^l_T] \leq w^i, \, \forall \, l = 1, ..., d \},$$

a proof similar to that of Theorem 2.4 still shows existence of Nash equilibria without need of Conditions 2.5 and 2.6.

Remark 2.8. *Theorem 2.4 still holds if we relax the condition of nonnegative costs and we allow the functions h^i and g^i to assume values in $\mathbb{R}$, but requiring, however, conditions ensuring that*

$$\inf_{V \in \mathcal{A}} \mathcal{J}^i(V, A^{-i}) > -\infty \quad \textit{for all} \quad A^{-i} \in \mathcal{A}^{N-1} \quad \textit{such that} \quad \mathbb{E}[|A^{-i}_T|] \leq \frac{2M}{\kappa} \vee K.$$

This allows also to apply the Theorem 2.4 in the case in which players aim at maximizing expected net profit functionals.

Remark 2.9 (Infinite Time-Horizon Case: $T = \infty$)**.** *Theorem 2.4 can be proved also in the case $T = \infty$. Indeed, we can consider the problem in which each player chooses a strategy in the set*

$$\mathcal{A}[0, \infty) = \left\{ A : [0, \infty) \times \Omega \to [0, \infty)^d \;\middle|\; \begin{array}{l} A \textit{ is an } \bar{\mathbb{F}}^{f,L}_+\textit{-adapted càdlàg process, with} \\ \textit{nondecreasing and nonnegative components} \end{array} \right\},$$

in order to minimize the cost functional

$$\mathcal{J}^i_\infty(A^i, A^{-i}) = \mathbb{E}\bigg[\int_0^\infty h^i(L_t, \mathbf{A}_t)\, dt + \int_{[0,\infty)} f^i_t \, dA^i_t \bigg].$$

Then, the arguments developed in the previous proof carry on upon replacing A_T with $A_\infty := \sup_{t \in [0,\infty)} A_t$.

3. The n-Lipschitz Game

In the notation of Section 2, for each $n \in \mathbb{N}$, define the space of *n-Lipschitz strategies*

$$\mathcal{L}(n) = \{ A \in \mathcal{A} \,|\, A \text{ is Lipschitz with Lipschitz constant smaller that } n \text{ and } A_0 = 0 \},$$

and the space of *n-Lipschitz profile strategies* as $\mathcal{L}^N(n) := \bigotimes_{i=1}^N \mathcal{L}(n)$. The set $\mathcal{L}(n)$ (resp. $\mathcal{L}^N(n)$) inherits from $\mathcal{A}$ (resp. $\mathcal{A}^N$) the order relation $\preccurlyeq$ (resp. $\preccurlyeq^N$) together with the associated lattice structure.

For each $n \in \mathbb{N}$, the set of n-Lipschitz profile strategies $\mathcal{L}^N(n)$, together with the cost functionals $\mathcal{J}^i$, define a game to which we will refer to as the *n-Lipschitz game*. We say that an n-Lipschitz profile strategy $\mathbf{A} \in \mathcal{L}^N(n)$ is a Nash equilibrium of the n-Lipschitz game if, for each $i = 1, ..., N$, we have $\mathcal{J}^i(\mathbf{A}) < \infty$ and

$$\mathcal{J}^i(A^i, A^{-i}) \leq \mathcal{J}^i(V^i, A^{-i}), \quad \text{for every} \quad V^i \in \mathcal{L}(n).$$

Theorem 3.1 (Existence of Nash Equilibria for the Submodular n-Lipschitz Game)**.** *Let Assumption 2.2 hold. Then, for each $n \in \mathbb{N}$, the set of Nash equilibria of the n-Lipschitz game $F \subset \mathcal{L}^N(n)$ is non empty, and the partially ordered set $(F, \preccurlyeq^N)$ is a complete lattice.*

Proof. The proof is organized in three steps.

(Step 1) The lattices $(\mathcal{L}^N(n), \preccurlyeq^N)$ and $(\mathcal{L}(n), \preccurlyeq)$ are complete.
With regards to *Step 1* in the proof of Theorem 2.4, we only have to show that the least upper bound and a greatest lower bound of any subset of $\mathcal{L}^N(n)$ still belongs to $\mathcal{L}^N(n)$. We will show now that for each set of indexes $\mathcal{I}$ and each subset $\{\mathbf{A}^j\}_{j\in\mathcal{I}}$ of $\mathcal{L}^N(n)$, its least upper bound $\mathbf{S}$ still lies in $\mathcal{L}^N(n)$. Analogous arguments apply to show that the greatest lower bound of any subset of $\mathcal{L}^N(n)$ is still in $\mathcal{L}^N(n)$.

Fix $q, \bar{q} \in Q$, with $Q := (\mathbb{Q} \cap [0,T]) \cap \{T\}$, such that $q > \bar{q}$, recall $\tilde{\mathbf{S}}$ from (2.8), and consider a countable subset $\mathcal{I}_q$ of $\mathcal{I}$ for which

$$\tilde{\mathbf{S}}_q = \sup_{h\in\mathcal{I}_q} \mathbf{A}^h_q.$$

We then have, $\mathbb{P}$-a.s.,

$$\tilde{\mathbf{S}}_q - \tilde{\mathbf{S}}_{\bar{q}} = \sup_{h\in\mathcal{I}_q} \left(\mathbf{A}^h_q - \operatorname*{ess\,sup}_{j\in\mathcal{I}} \mathbf{A}^j_{\bar{q}} \right) \leq \sup_{h\in\mathcal{I}_q} \left(\mathbf{A}^h_q - \mathbf{A}^h_{\bar{q}} \right) \leq n\,|q - \bar{q}|,$$

and, since $\tilde{\mathbf{S}}$ in nondecreasing, we conclude that $|\tilde{\mathbf{S}}_q - \tilde{\mathbf{S}}_{\bar{q}}| \leq n\,|q - \bar{q}|$, $\mathbb{P}$-a.s. Therefore, since Q is countable, recalling the definition of $\mathbf{S}$ given in (2.10), we deduce that $\mathbf{S}$ is Lipschitz continuous with Lipschitz constant bounded by n.

(Step 2) The best reply maps $R^i : \mathcal{L}^N(n) \to \mathcal{L}(n)$ (cf. (2.3)) are well defined.
Fix i and take $\mathbf{A} \in \mathcal{L}^N(n)$. We apply the classical direct method to find $\bar{V} \in \mathcal{L}(n)$ such that

$$\mathcal{J}^i(\bar{V}, A^{-i}) = \min_{V\in\mathcal{L}(n)} \mathcal{J}^i(V, A^{-i}).$$

Take a minimizing sequence $\{V^j\}_{j\in\mathbb{N}} \subset \mathcal{L}(n)$. Since, for each $j \in \mathbb{N}$, we have that V^j is Lipschitz, we can define $\mathbb{P}\otimes dt$-a.e. the time derivative of V^j; that is, the $\mathbb{F}$-progressively measurable $\mathbb{R}^d$-valued process $v^j_t := dV^j_t/dt$. Since the sequence $\{v^j\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{L}^2(\Omega\times[0,T];\mathbb{R}^d)$ (as any of its elements is bounded by n), by Banach-Saks' theorem (see, e.g., p. 314 of [62]) we can extract a subsequence (still denoted by j) whose Cesàro sums $\{c^j\}_{j\in\mathbb{N}}$ converge strongly in $\mathbb{L}^2(\Omega\times[0,T];\mathbb{R}^d)$ to some $\bar{\mathbb{F}}^{f,L}_+$-progressively measurable $\bar{v} \in \mathbb{L}^2(\Omega\times[0,T];\mathbb{R}^d)$. By passing to another subsequence $\{c^{j_m}\}_{m\in\mathbb{N}}$ we can assume that $\{c^{j_m}\}_{m\in\mathbb{N}}$ converges $\mathbb{P}\otimes dt$-a.e. to $\bar{v}$, which allows to conclude that $\bar{v}_t \in [0,n]^d$, $\mathbb{P}\otimes dt$-a.e.

Define then $\bar{V}_t := \int_0^t \bar{v}_s\,ds$, and observe that, by the properties of $\bar{v}$, we have $\bar{V} \in \mathcal{L}(n)$. Also, $\mathbb{P}$-a.s., $\int_0^t c^{j_m}_s\,ds$ converges to $\bar{V}_t$ for each $t \in [0,T]$, and the convexity of $\mathcal{J}^i(\cdot, A^{-i})$ guarantees that the sequence $\{\int_0^\cdot c^{j_m}_s\,ds\}_{j\in\mathbb{N}}$ is still minimizing. Hence, thanks to the lower semi-continuity and the convexity of h^i and g^i, and to Fatou's lemma, we can conclude that

$$\mathcal{J}^i(\bar{V}, A^{-i}) = \mathcal{J}^i(\lim_m \textstyle\int_0^\cdot c^{j_m}, A^{-i}) \leq \liminf_m \mathcal{J}^i(\textstyle\int_0^\cdot c^{j_m}, A^{-i}) = \min_{V\in\mathcal{L}(n)} \mathcal{J}^i(V, A^{-i}).$$

The latter yields that $\bar{V}$ minimizes $\mathcal{J}^i(\cdot, A^{-i})$. In fact, $\bar{V}$ is the unique minimizer of $\mathcal{J}^i(\cdot, A^{-i})$ by strict convexity of the costs.

(Step 3) Existence of Nash equilibria.

By employing arguments as those in *Step 3* of the proof of Theorem 2.4 we can deduce that the best reply map $\mathbf{R} = (R^1, ..., R^N) : \mathcal{L}^N(n) \to \mathcal{L}^N(n)$ is monotone increasing in the complete lattice $(\mathcal{L}^N(n), \preccurlyeq^N)$. Then, the thesis of the theorem follows from Tarski's fixed point theorem. $\square$

4. Existence and Approximation of Weak Nash Equilibria in the Submodular Monotone-Follower Game

In this section we will investigate connections between the monotone-follower game and the n-Lipschitz games.

4.1. **Weak Formulation of the Monotone-Follower Game.** For $T \in (0, \infty)$ and an arbitrary $m \in \mathbb{N}$, we introduce the following measurable spaces:

- $\mathcal{C}_+^m$ denotes the set of $\mathbb{R}^m$-valued continuous function on $[0, T]$ with nonnegative components, endowed with the Borel σ-algebra generated by the uniform convergence norm;
- $\mathcal{D}^m$ denotes the Skorokhod space of $\mathbb{R}^m$-valued càdlàg functions, defined on $[0, T]$, endowed with the Borel σ-algebra generated by the Skorokhod topology;
- $\mathcal{D}_\uparrow^m$ denotes the Skorokhod space of $\mathbb{R}^m$-valued nondecreasing, nonnegative càdlàg functions, defined on $[0, T]$, endowed with the Borel σ-algebra generated by the Skorokhod topology.

We refer to Chapter 3 in [13] for more details on the Skorokhod space. Also, let $\mathcal{P}(\mathcal{C}_+^m)$, $\mathcal{P}(\mathcal{D}^m)$ and $\mathcal{P}(\mathcal{D}_\uparrow^m)$ denote the set of probability measures on the Borel σ-algebras of $\mathcal{C}_+^m$, $\mathcal{D}^m$ and $\mathcal{D}_\uparrow^m$, respectively. Finally, denote by $\mathcal{P}(\mathcal{C}_+^m \times \mathcal{D}^m \times \mathcal{D}_\uparrow^m)$ the set of probability measures on the product σ-algebra.

Moreover, denote by $(\pi_f, \pi_L) : \mathcal{C}_+^{Nd} \times \mathcal{D}^k \times [0, T] \to \mathbb{R}^{Nd+k}$ the canonical projection, i.e., set $(\pi_f, \pi_L)_t(f, L) = (f_t, L_t)$ for each $(f, L) \in \mathcal{C}_+^{Nd} \times \mathcal{D}^k$ and $t \in [0, T]$. Also, for a probability measure $\mathbb{P} \in \mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k)$, denote by $\bar{\mathbb{F}}_+^{\pi_f, \pi_L}$ the right continuous extension of the filtration on $\mathcal{C}_+^{Nd} \times \mathcal{D}^k$ generated by the canonical projections π_f and π_L, augmented by the $\mathbb{P}$-null sets.

We now give a weak formulation of the monotone-follower game. Assume to be given a distribution $\mathbb{P}_0 \in \mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k)$ such that the projection process $\pi_f : \mathcal{C}_+^{Nd} \times \mathcal{D}^k \times [0, T] \to \mathbb{R}^{Nd}$ is a semimartingale with respect to the filtration $\bar{\mathbb{F}}_+^{\pi_f, \pi_L}$.

Definition 2. *We call a basis a 5-tuple $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$ such that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, L is an $\mathbb{R}^k$-valued càdlàg process, $f = (f^1, ..., f^N)$ is an $\mathbb{R}^{Nd}$-valued continuous, nonnegative semimartingale with respect to the filtration $\bar{\mathbb{F}}_+^{f,L}$, and $\mathbb{P} \circ (f, L)^{-1} = \mathbb{P}_0$.*

For each basis β, we then give the relative notion of admissible strategy.

Definition 3. *Given a basis $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$, an* admissible strategy associated to β *is an $\mathbb{R}^d$-valued càdlàg, nondecreasing, nonnegative process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{A}_\beta$ the set of admissible strategies associated to the basis β. Moreover, we define the space of admissible profile strategies associated to the basis β as $\mathcal{A}_\beta^N := \bigotimes_{i=1}^N \mathcal{A}_\beta$.*

Given a basis $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$, for each $i \in \{1, ..., N\}$ and each admissible strategy $A^i \in \mathcal{A}_\beta$ we define the cost functionals

$$\mathcal{J}_\beta^i(A^i, A^{-i}) := \mathbb{E}^{\mathbb{P}}[C^i(f, L, \mathbf{A})] = \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T h^i(L_t, \mathbf{A}_t)\, dt + g^i(L_T, \mathbf{A}_T) + \int_{[0,T]} f_t^i \, dA_t^i\bigg], \tag{4.1}$$

where $A^{-i} := (A^j)_{j \neq i}$, $\mathbf{A} := (A^i, A^{-i})$ and $\mathbb{E}^{\mathbb{P}}$ denotes the expectation under the probability measure $\mathbb{P}$.

We finally introduce a notion of equilibrium that we will refer to as *weak Nash equilibrium.*

Definition 4 (Weak Nash Equilibrium)**.** *Given a basis $\bar\beta$ and an admissible profile strategy $\bar{\mathbf{A}} \in \mathcal{A}^N_{\bar\beta}$, we say that the couple $(\bar\beta, \bar{\mathbf{A}})$ is a weak Nash equilibrium if, for every $i = 1, ..., N$, we have*

$$\mathcal{J}^i_{\bar\beta}(\bar A^i, \bar A^{-i}) \le \mathcal{J}^i_{\bar\beta}(V^i, \bar A^{-i}), \quad \textit{for every} \quad V^i \in \mathcal{A}_{\bar\beta}.$$

4.2. **Assumptions and Preliminary Lemmata.** In this subsection we specify the main assumptions of this section, we introduce some notations, and we provide some preliminary lemmata.

Assumption 4.1. *Let Assumption 2.2 hold and, for each $i = 1, ..., N$, assume that*

(1) g^i and h^i are continuous and continuously differentiable in the variable $a^i \in \mathbb{R}^d$;

(2) there exist $\gamma_1, \gamma_2 > 1$ such that the d-dimensional gradients $\nabla_i h^i$ and $\nabla_i g^i$ of the functions h^i and g^i with respect to the (d-dimensional) variable a^i satisfy

$$|\nabla_i h^i(l, a)| + |\nabla_i g^i(l, a)| \le C(1 + |l|^{\gamma_1} + |a|^{\gamma_2}), \tag{4.2}$$

for each $l \in \mathbb{R}^k$ and $a = (a^1, ..., a^N) \in \mathbb{R}^{Nd}$.

Moreover, there exist measurable functions $H^i, G^i : \mathbb{R}^k \to \mathbb{R}$ such that $h^i(l, 0, a^{-i}) \le H^i(l)$ and $g^i(l, 0, a^{-i}) \le G^i(l)$, with

$$\mathbb{E}^{\mathbb{P}_0}\left[\int_0^T |H^i((\pi_L)_s)|^q\, ds + |G^i((\pi_L)_T)|^q\right] < \infty \tag{4.3}$$

and

$$\mathbb{E}^{\mathbb{P}_0}\left[\sup_{s\in[0,T]} \left(|(\pi_L)_s|^{\alpha\gamma_1 p} + |(\pi_f)_s|^{\alpha p}\right)\right] < \infty, \tag{4.4}$$

where $q := \alpha \max\{\gamma_2\, p,\, p/(p-1)\}$ for some $p, \alpha > 1$;

(3) there exists a constant $c > 0$ such that

$$\mathbb{P}_0\left[(\pi_f)^i_t \ge c, \,\forall\, t \in [0,T], \,\forall\, i = 1, ..., N\right] = 1, \tag{4.5}$$

and the total conditional variation (see definition (B.3) in the Appendix B) of π_L over the interval $[0, T]$ is finite; that is, $V^{\mathbb{P}_0}_T(\pi_L) < \infty$.

The following lemma will be useful in our subsequent analysis. It exploits the convexity of h^i and g^i in order to obtain a subgradient inequality. The arguments of its proof are similar to those already employed in [9], [30] and [65], but we provide a proof in Appendix A for the sake of completeness.

Lemma 4.2. *Let $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$ be a basis and take a profile strategy $\mathbf{A} = (A^1, ..., A^N) \in \mathcal{A}^N_\beta$ such that $\mathcal{J}^i(\mathbf{A}) < \infty$ for $i = 1, ..., N$. For each i and each $B^i \in \mathcal{A}_\beta$ we have*

$$\mathcal{J}^i_\beta(B^i, A^{-i}) - \mathcal{J}^i_\beta(A^i, A^{-i}) \ge \mathbb{E}^{\mathbb{P}}\left[\int_{[0,T]} Y^i_t(dB^i_t - dA^i_t)\right]$$

where we define the (non adapted) process

$$Y^i_t := \int_t^T \nabla_i h^i(L_t, A^i_t, A^{-i}_t)\, dt + \nabla_i g^i(L_T, A^i_T, A^{-i}_T) + f^i_t, \quad t \in [0, T]. \tag{4.6}$$

Fix a basis $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$ and denote by $\bar{\mathbb{F}}^{f,L}_+ = \{\bar{\mathcal{F}}^{f,L}_{t+}\}_{t\in[0,T]}$ the right-continuous extension of the filtration generated by f and L, augmented by the $\mathbb{P}$-null sets. For each $n \in \mathbb{N}$, consider a Nash equilibrium $\mathbf{A}^n = (A^{1,n}, ..., A^{N,n})$ of the n-Lipschitz game as in Theorem 3.1. The next lemma shows that any Nash equilibria of the n-Lipschtz game satisfy certain *first order conditions*. Its proof follows arguments analogus to that in the proof of Proposition 27 in [47], and it is postponed to Appendix A.

Lemma 4.3. *Define, for every $n \in \mathbb{N}$ and every $i = 1, ..., N$, the (non adapted) continuous process*

$$Y_t^{i,n} := \int_t^T \nabla_i h^i(L_t, \mathbf{A}_t^n)\, dt + \nabla_i g^i(L_T, \mathbf{A}_T^n) + f_t^i, \quad t \in [0,T], \quad \mathbb{P}-a.s. \tag{4.7}$$

Then, under Assumption 4.1, we have

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t^{i,n} dA_t^{i,n}\right] = -n\, \mathbb{E}^{\mathbb{P}}\left[\int_0^T (Y_t^{i,n})^- \,\mathbf{1}\, dt\right] \tag{4.8}$$

where $\mathbf{1} := (1, ..., 1) \in \mathbb{R}^d$, and

$$\lim_n \mathbb{E}^{\mathbb{P}}\left[\int_0^T (Y_t^{i,n})^-\, dt\right] = 0. \tag{4.9}$$

4.3. **Existence and Approximation of Weak Nash Equilibria.** We now state and prove the main result of this section.

For an arbitrary $m \in \mathbb{N}$, consider on the space $\mathcal{C}_+^m$ the topology given by the convergence in the uniform norm. Furthermore, on the space $\mathcal{D}^m$ consider the *pseudopath topology* τ_{pp}^T; that is, the topology on $\mathcal{D}^m$ induced by the convergence in the measure $dt + \delta_T$ on the interval $[0,T]$, where dt denotes the Lebesgue measure, and δ_T denotes the Dirac measure at the terminal time T. The space $\mathcal{D}_\uparrow^m$ is a closed subset of the topological space $(\mathcal{D}^m, \tau_{pp}^T)$, and the Borel σ-algebra induced by the topology τ_{pp}^T, coincides with the σ-algebra induced by the Skorokhod topology (see also the Appendix in [47]). Notice that the topological spaces $(\mathcal{D}^m, \tau_{pp}^T)$ and $(\mathcal{D}_\uparrow^m, \tau_{pp}^T)$ are separable, but not Polish (see, e.g., [54]). Finally, on the product space $\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd}$, consider the product topology, and on $\mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$ consider the topology of weak convergence of probability measures.

Fix a basis $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$ and consider, for each $n \in \mathbb{N}$, a Nash equilibrium $\mathbf{A}^n = (A^{1,n}, ..., A^{N,n})$ of the n-Lipschitz game as in Theorem 3.1. Define, for $n \in \mathbb{N}$, the law $\mathbb{P}^n := \mathbb{P} \circ (f, L, \mathbf{A}^n)^{-1}$ in $\mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$; with a slight abuse of terminology, we will refer to the law $\mathbb{P}^n$ as the law of the Nash equilibrium $\mathbf{A}^n$. We then have the following theorem.

Theorem 4.4. *Under Assumption 4.1 the following statements hold.*

(1) The sequence $\{\mathbb{P}^n\}_{n\in\mathbb{N}}$ of the laws of the Nash equilibria of the n-Lipschitz games is weakly relatively compact in $\mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$.

(2) Any accumulation point $\bar{\mathbb{P}}$ is the law of a weak Nash equilibrium of the monotone-follower game; that is, there exist a basis $\bar\beta = (\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{Q}}, \bar f, \bar L)$ and an admissible profile strategy $\bar{\mathbf{A}} \in \mathcal{A}_{\bar\beta}^N$, such that $(\bar\beta, \bar{\mathbf{A}})$ is a weak Nash equilibrium of the monotone-follower game and $\bar{\mathbb{P}} = \bar{\mathbb{Q}} \circ (\bar f, \bar L, \bar{\mathbf{A}})^{-1}$.

Proof. We prove the two claims of the theorem separately.

Proof of Claim 1. By assumption we have $V_T^{\mathbb{P}}(L) < \infty$, and by Lemma A.1 in the Appendix A we have

$$\sup_n \mathbb{E}^{\mathbb{P}}\left[|\mathbf{A}_T^n|^q\right] < \infty, \tag{4.10}$$

where $q > 1$ is as in Assumption 4.1. Therefore, from Lemma B.2, we can deduce that the sequence $\{\mathbf{A}^n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{D}_\uparrow^{Nd})$, and that L in tight in $\mathcal{P}(\mathcal{D}^k)$. Moreover, since the space $\mathcal{C}_+^{Nd}$ is Polish, $\mathbb{P} \circ f^{-1}$ is regular, and hence f is tight in $\mathcal{P}(\mathcal{C}_+^{Nd})$ (see, e.g., Remark 13.27 at p. 260 in [43]). This implies that the sequence $\{(f, L, \mathbf{A}^n)\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$.

By Prokhorov's theorem (see, e.g., Theorem 13.29 at p. 261 in [43]), there exists a subsequence of indexes (still denoted by n) and a probability measure $\bar{\mathbb{P}} \in \mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$ such that the sequence $\mathbb{P}^n$ converges weakly to $\bar{\mathbb{P}}$. The first claim of the theorem is thus proved.

Proof of Claim 2. Thanks to an extension of Skorokhod's theorem for separable spaces (see Theorem 3 in [28]), there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{Q}})$, and, on it, a sequence

$$\{(\bar{f}^n, \bar{L}^n, \bar{\mathbf{A}}^n)\}_{n\in\mathbb{N}}$$

of $\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd}$-valued random variables, and a $\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd}$-valued random variable $(\bar{f}, \bar{L}, \bar{\mathbf{A}})$, such that

(i) $\bar{\mathbb{Q}} \circ (\bar{f}^n, \bar{L}^n, \bar{\mathbf{A}}^n)^{-1} = \mathbb{P}^n$ and $\bar{\mathbb{Q}} \circ (\bar{f}, \bar{L}, \bar{\mathbf{A}})^{-1} = \bar{\mathbb{P}}$;
(ii) for almost all $\omega \in \bar{\Omega}$, we have

$$\sup_{t\in[0,T]} |\bar{f}_t^n(\omega) - \bar{f}_t(\omega)| \to 0, \tag{4.11}$$

as well as

$$(\bar{L}^n(\omega), \bar{\mathbf{A}}^n(\omega)) \to (\bar{L}(\omega), \bar{\mathbf{A}}(\omega)) \quad \text{in the Lebesgue measure } dt \text{ on } [0,T], \tag{4.12}$$

and

$$(\bar{L}_T^n(\omega), \bar{\mathbf{A}}_T^n(\omega)) \to (\bar{L}_T(\omega), \bar{\mathbf{A}}_T(\omega)). \tag{4.13}$$

Define then $\bar{\beta} := (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{Q}}, \bar{f}, \bar{L})$. Since $\mathbb{P} \circ (f, L)^{-1}$ is constantly $\mathbb{P}_0$, then the same holds for its limit; that is, $\bar{\mathbb{Q}} \circ (\bar{f}, \bar{L})^{-1} = \mathbb{P}_0$, and this implies that $\bar{\beta}$ is a basis.

Now, for every $i = 1, ..., N$, we define on the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{Q}})$ the sequence of continuous processes $\{\bar{Y}^{i,n}\}_{n\in\mathbb{N}}$ by

$$\bar{Y}_t^{i,n} := \int_t^T \nabla_i h^i(\bar{L}_t^n, \bar{\mathbf{A}}_t^n)\, dt + \nabla_i g^i(\bar{L}_T^n, \bar{\mathbf{A}}_T^n) + \bar{f}_t^{i,n} \quad \text{for} \quad t \in [0,T], \tag{4.14}$$

as well as the process

$$\bar{Y}_t^i := \int_t^T \nabla_i h^i(\bar{L}_t, \bar{\mathbf{A}}_t)\, dt + \nabla_i g^i(\bar{L}_T, \bar{\mathbf{A}}_T) + \bar{f}_t^i \quad \text{for} \quad t \in [0,T]. \tag{4.15}$$

The following claims summarize two key properties of the processes $\bar{Y}^i$ and $\bar{\mathbf{A}}$ that will guarantee that $(\bar{\beta}, \bar{\mathbf{A}})$ is a weak Nash equilibrium as in Definition 4.

For every $i = 1, ..., N$, we now prove that the following hold $\bar{\mathbb{Q}}$-a.s.:

(2.a) $\bar{Y}_t^i \geq 0$ for every $t \in [0,T]$;
(2.b) $\int_{[0,T]} \bar{Y}_t^i \, d\bar{A}_t^i = 0$.

(Proof of 2.a) We begin by proving that $\bar{Y}^n \to \bar{Y}$ in $\mathbb{L}^1(\bar{\mathbb{Q}}\otimes dt)$. For $i = 1, ..., N$, from Lemma A.3 (see Appendix) we have that $\bar{\mathbb{Q}} \otimes dt$-a.e. $\bar{Y}^{i,n}$ converges to $\bar{Y}^i$. Moreover, for $p > 1$ as in Assumption 4.1, by the growth condition (4.2) we have that

$$
\begin{aligned}
\mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\sup_{t\in[0,T]} |\bar{Y}^{i,n}_t|^p\bigg] &\le \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\sup_{t\in[0,T]} \bigg|\int_t^T \nabla_i h^i(\bar{L}^n_t, \bar{\mathbf{A}}^n_t)\, dt + \nabla_i g^i(\bar{L}^n_T, \bar{\mathbf{A}}^n_T) + \bar{f}^{i,n}_t\bigg|^p\bigg] \\
&\le \widetilde{C}\, \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\sup_{t\in[0,T]} \Big(1 + |\bar{L}^n_t|^{\gamma_1 p} + |\bar{\mathbf{A}}^n_t|^{\gamma_2 p} + |\bar{f}^{i,n}_t|^p\Big)\bigg] \\
&\le \widetilde{C}\, \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[1 + |\bar{\mathbf{A}}^n_T|^{\gamma_2 p} + \sup_{t\in[0,T]} \Big(|\bar{L}^n_t|^{\gamma_1 p} + |\bar{f}^{i,n}_t|^p\Big)\bigg] \\
&\le \widetilde{C}\bigg(1 + \mathbb{E}^{\mathbb{P}}[|\mathbf{A}^n_T|^{\gamma_2 p}] + \mathbb{E}^{\mathbb{P}_0}\bigg[\sup_{t\in[0,T]} \Big(|(\pi_L)_t|^{\gamma_1 p} + |(\pi_f)^i_t|^p\Big)\bigg]\bigg),
\end{aligned} \tag{4.16}
$$

where $\widetilde{C}$ is a constant that may vary from line to line. Using then the integrability condition (4.4) in Assumption 4.1 and the estimates (4.10) (recall that by assumption $\gamma_2 p < q$), we have

$$
\sup_n \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\sup_{t\in[0,T]} |\bar{Y}^{i,n}_t|^p\bigg] < \infty, \tag{4.17}
$$

which implies that the sequence $\bar{Y}^{i,n}$ is uniformly integrable. From Theorem 6.25 at p. in [43], we deduce then that $\bar{Y}^n \to \bar{Y}$ in $\mathbb{L}^1(\bar{\mathbb{Q}} \otimes dt)$. Now, from (4.9) in Lemma 4.3, we easily find

$$
0 = \lim_n \mathbb{E}^{\mathbb{P}}\left[\int_0^T (Y^{i,n}_t)^- \, dt\right] = \lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T (\bar{Y}^{i,n}_t)^- \, dt\right] = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T (\bar{Y}^{i}_t)^- \, dt\right],
$$

and by continuity of $\bar{Y}^i$ we conclude that $\bar{\mathbb{Q}}$-a.s.

$$
\bar{Y}^i_t \ge 0\,,\ \forall\, t \in [0,T],\ \forall\, i = 1, ..., N. \tag{4.18}
$$

(Proof of 2.b) Observe first that, by the convergence at the terminal point (4.13) together with Fatou's lemma and the estimate (4.10) we have

$$
\mathbb{E}^{\bar{\mathbb{Q}}}[|\bar{\mathbf{A}}_T|^q] \le \sup_n \mathbb{E}^{\bar{\mathbb{Q}}}[|\bar{\mathbf{A}}^n_T|^q] = \sup_n \mathbb{E}^{\mathbb{P}}[|\mathbf{A}^n_T|^q] < \infty. \tag{4.19}
$$

Furthermore, computations analogous to those employed in (4.16) yield

$$
\mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\sup_{t\in[0,T]} |\bar{Y}^{i,n}_t|^{\alpha p}\bigg] \le \widetilde{C}\bigg(1 + \mathbb{E}^{\mathbb{P}}[|\mathbf{A}^n_T|^{\alpha\gamma_2 p}] + \mathbb{E}^{\mathbb{P}_0}\bigg[\sup_{t\in[0,T]} \Big(|(\pi_L)_t|^{\alpha\gamma_1 p} + |(\pi_f)^i_t|^{\alpha p}\Big)\bigg]\bigg), \tag{4.20}
$$

as well as,

$$
\mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\sup_{t\in[0,T]} |\bar{Y}^{i}_t|^{\alpha p}\bigg] \le \widetilde{C}\bigg(1 + \mathbb{E}^{\bar{\mathbb{Q}}}[|\bar{\mathbf{A}}_T|^{\alpha\gamma_2 p}] + \mathbb{E}^{\mathbb{P}_0}\bigg[\sup_{t\in[0,T]} \Big(|(\pi_L)_t|^{\alpha\gamma_1 p} + |(\pi_f)^i_t|^{\alpha p}\Big)\bigg]\bigg). \tag{4.21}
$$

Now, the estimates (4.10), (4.19), (4.20) and (4.21) implies that

$$
\sup_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\sup_{t\in[0,T]} |\bar{Y}^{i,n}_t|^{\alpha p} + \sup_{t\in[0,T]} |\bar{Y}^{i}_t|^{\alpha p} + |\bar{\mathbf{A}}^n_T|^{\frac{\alpha p}{p-1}} + |\bar{\mathbf{A}}_T|^{\frac{\alpha p}{p-1}}\right] < \infty,
$$

which, together with the convergence established in Lemma A.3 in Appendix A, allows us to use Lemma B.3 in Appendix B in order to deduce that

$$
\mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]} \bar{Y}^i_t \, d\bar{A}^i_t\bigg] = \lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]} \bar{Y}^{i,n}_t \, d\bar{A}^{i,n}_t\bigg]. \tag{4.22}
$$

Furthermore, since for each $n \in \mathbb{N}$ we have $\bar{A}_0^{i,n} = 0$ $\bar{\mathbb{Q}}$-a.s., thanks to (4.8) in Lemma 4.3 we have that

$$\mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]} \bar{Y}_t^{i,n_j}\, d\bar{A}_t^{i,n_j}\bigg] = \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_0^T \bar{Y}_t^{i,n_j}\, d\bar{A}_t^{i,n_j}\bigg] = \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T Y_t^{i,n_j}\, dA_t^{i,n_j}\bigg] \leq 0,$$

hence, due to (4.22), we deduce that

$$\mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]} \bar{Y}_t^i d\bar{A}_t^i\bigg] \leq 0.$$

This implies, thanks to the non negativity of $\bar{Y}^i$ established in *(2.a)*, that $\bar{\mathbb{Q}}$-a.s.

$$\int_{[0,T]} \bar{Y}_t^i d\bar{A}_t^i = 0,$$

so that *(2.b)* is proved.

It does remain to conclude that the couple $(\bar{\beta}, \bar{\mathbf{A}})$ is a weak Nash equilibrium of the game. Fix $i \in \{1, ..., N\}$, and consider an admissible strategy $B^i \in \mathcal{A}(\bar{\beta})$. By Lemma 4.2 and Claims $(2.a)$ and $(2.b)$ we have

$$\mathcal{J}_{\bar{\beta}}^i(B^i, \bar{A}^{-i}) - \mathcal{J}_{\bar{\beta}}^i(\bar{A}^i, \bar{A}^{-i}) \geq \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]} \bar{Y}_t^i(dB_t^i - d\bar{A}_t^i)\bigg] = \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]} \bar{Y}_t^i dB_t^i\bigg] \geq 0,$$

which in fact completes the proof. □

4.4. **On Lipschitz ε-Nash Equilibria for the Monotone-Follower Game.** In this subsection we prove another connection between the Lipschitz games and the monotone-follower game by showing that ε-Nash equilibria of the monotone-follower game can be realized as Nash equilibria of the n-Lipschitz game, for n sufficiently large.

As in Subsection 4.3, in the following we consider fixed a basis $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$, and, for each $n \in \mathbb{N}$, let $\mathbf{A}^n = (A^{1,n}, ..., A^{N,n})$ be a Nash equilibrium of the n-Lipschitz game as in Theorem 3.1.

Theorem 4.5. *Suppose that Assumption 4.1 holds and that there exists a constant $C > 0$ such that*

$$|h^i(l, a)| + |g^i(l, a)| \leq C(1 + |l|^{\gamma_1} + |a^{-i}|^{\gamma_2}), \tag{4.23}$$

for each $l \in \mathbb{R}^k$ and $a = (a^1, ..., a^N) \in \mathbb{R}^{Nd}$.

Then, for each $\varepsilon > 0$, there exists n_ε such that the Nash equilibrium $\mathbf{A}^{n_\varepsilon}$ of the n_ε-Lipschitz game is an ε-Nash equilibrium of the monotone-follower game; that is, for each $i = 1, ..., N$

$$\mathcal{J}_\beta^i(A^{i,n_\varepsilon}, A^{-i,n_\varepsilon}) \leq \mathcal{J}_\beta^i(B^i, A^{-i,n_\varepsilon}) + \varepsilon \quad \textit{for each} \quad B^i \in \mathcal{A}_\beta.$$

Proof. We argue by contraddiction and we suppose that the thesis is false. Then, there exists $\varepsilon > 0$ such that, for each $n \in \mathbb{N}$, there exist $i_n \in \{1, ..., N\}$ and an admissible strategy $B^n \in \mathcal{A}_\beta$ with

$$\mathcal{J}_\beta^{i_n}(\mathbf{A}^n) > \mathcal{J}_\beta^{i_n}(B^n, A^{-i_n,n}) + \varepsilon.$$

Since the number of indexes of the players is finite, we can suppose that there exists $i \in \{1, ..., N\}$ such that, for each $n \in \mathbb{N}$,

$$\mathcal{J}_\beta^i(\mathbf{A}^n) > \mathcal{J}_\beta^i(B^n, A^{-i,n}) + \varepsilon. \tag{4.24}$$

Recall now that, for each $n \in \mathbb{N}$, $\mathbf{A}^n$ is a Nash equilibrium for the n-Lipschitz game and notice that the process constantly equal to zero is admissible. Hence, from (4.24), and using the coercivity condition (4.5) and the integrability condition (4.3) in Assumption 4.1, we find

$$c\,\mathbb{E}^{\mathbb{P}}[|B^n_T|] \le \mathcal{J}^i_\beta(B^n, A^{-i,n}) < \mathcal{J}^i_\beta(\mathbf{A}^n) - \varepsilon$$
$$\le \mathcal{J}^i_\beta(0, A^{-i,n}) \le \mathbb{E}^{\mathbb{P}_0}\left[\int_0^T H^i((\pi_L)_t)\,dt + G^i((\pi_L)_T)\right] < \infty,$$

which implies that

$$\sup_n \mathbb{E}^{\mathbb{P}}[|B^n_T|] < \infty. \tag{4.25}$$

With arguments analogous to those employed in the proof of *Claim 1* of Theorem 4.4, from the tightness condition (4.25) we deduce that there exists a subsequence of indexes (still denoted by n) and a probability measure $\tilde{\mathbb{P}} \in \mathcal{P}(\mathcal{C}^{Nd}_+ \times \mathcal{D}^k \times \mathcal{D}^{(1+N)d}_\uparrow)$ such that the sequence $\mathbb{P} \circ (f, L, B^n, \mathbf{A}^n)^{-1}$ converges weakly to $\tilde{\mathbb{P}}$.

Then, thanks again to an extension of Skorokhod's theorem (see Theorem 3 in [28]), there exists a probability space $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{Q}})$, and, on it, a sequence

$$\{(\bar f^n, \bar L^n, \bar B^n, \bar{\mathbf{A}}^n)\}_{n\in\mathbb{N}}$$

of $\mathcal{C}^{Nd}_+ \times \mathcal{D}^k \times \mathcal{D}^{(1+N)d}_\uparrow$-valued random variables, and a $\mathcal{C}^{Nd}_+ \times \mathcal{D}^k \times \mathcal{D}^{(1+N)d}_\uparrow$-valued random variable $(\bar f, \bar L, \bar B, \bar{\mathbf{A}})$, such that

(i) $\bar{\mathbb{Q}} \circ (\bar f^n, \bar L^n, \bar B^n, \bar{\mathbf{A}}^n)^{-1} = \tilde{\mathbb{P}}^n$ and $\bar{\mathbb{Q}} \circ (\bar f, \bar L, \bar B, \bar{\mathbf{A}})^{-1} = \tilde{\mathbb{P}}$;
(ii) for $\bar{\mathbb{Q}}$-almost all $\omega \in \bar\Omega$, we have

$$\sup_{t\in[0,T]} |\bar f^n_t(\omega) - \bar f_t(\omega)| \to 0, \tag{4.26}$$

as well as

$$(\bar L^n(\omega), \bar B^n(\omega), \bar{\mathbf{A}}^n(\omega)) \to (\bar L(\omega), \bar B(\omega), \bar{\mathbf{A}}(\omega)) \tag{4.27}$$

in the Lebesgue measure dt on $[0,T]$, and

$$(\bar L^n_T(\omega), \bar B^n_T(\omega), \bar{\mathbf{A}}^n_T(\omega)) \to (\bar L_T(\omega), \bar B_T(\omega), \bar{\mathbf{A}}_T(\omega)). \tag{4.28}$$

Moreover, as in Lemma A.2, we can deduce that, for $\bar{\mathbb{Q}}$-almost all $\omega \in \bar\Omega$, there exists a constant $M(\omega) < \infty$ such that

$$\sup_n \sup_{t\in[0,T]} (|\bar L^n_t(\omega)| + |\bar{\mathbf{A}}^n_t(\omega)| + |\bar L_t(\omega)| + |\bar{\mathbf{A}}_t(\omega)|) \le M(\omega).$$

Hence, for $\bar{\mathbb{Q}}$-almost all $\omega \in \Omega$, we can find, by continuity of h^i, another constant $K(\omega)$ such that

$$\sup_n \sup_{t\in[0,T]} \Big[h^i(\bar L^n_t(\omega), \bar B^n_t(\omega), \bar A^{-i,n}_t(\omega)) + h^i(\bar L_t(\omega), \bar B_t(\omega), \bar A^{-i}_t(\omega))\Big] \le K(\omega),$$

and, by the convergence established in (4.27) and in (4.28), we conclude that $\bar{\mathbb{Q}}$-a.s.

$$\begin{aligned}\lim_n \int_0^T h^i(\bar L^n_t, \bar B^n_t, \bar A^{-i,n}_t)\,dt + g^i(\bar L^n_T, \bar B^n_T, \bar A^{-i,n}_T)& \\ = \int_0^T h^i(\bar L_t, \bar B_t, , \bar A^{-i}_t)\,dt + g^i(\bar L_T, \bar B_T, \bar A^{-i}_T),&\end{aligned} \tag{4.29}$$

where we have also used that h^i and g^i are continuous. Furthermore, thanks to the growth condition (4.23), for $p > 1$ as in Assumption 4.1, we can find a suitable constant $\tilde C > 0$ such that

$$\sup_n \mathbb{E}^{\bar{\mathbb{Q}}}\Big[\Big|\int_0^T h^i(\bar{L}_t^n, \bar{B}_t^n, \bar{A}_t^{-i,n})\,dt + g^i(\bar{L}_T^n, \bar{B}_T^n, \bar{A}_T^{-i,n})\Big|^p\Big] \le \tilde{C}\sup_n\left(1 + \mathbb{E}^{\mathbb{P}_0}\left[\sup_{t\in[0,T]}|(\pi_L)_t|^{\gamma_1 p}\right] + \mathbb{E}^{\mathbb{P}}\left[|\mathbf{A}_T^n|^{\gamma_2 p}\right]\right) < \infty, \tag{4.30}$$

where the integrability of the right-hand side follows from Condition (4.4) and from Lemma A.1 in Appendix A. Finally, the limit in (4.29), together with the uniform integrability in (4.30), allows us to conclude that

$$\lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T h^i(\bar{L}_t^n, \bar{B}_t^n, \bar{A}_t^{-i,n})\,dt + g^i(\bar{L}_T^n, \bar{B}_T^n, \bar{A}_T^{-i,n})\right] = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T h^i(\bar{L}_t, \bar{B}_t, \bar{A}_t^{-i})\,dt + g^i(\bar{L}_T, \bar{B}_T, \bar{A}_T^{-i})\right]. \tag{4.31}$$

With a similar reasoning we also find

$$\lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T h^i(\bar{L}_t^n, \bar{A}_t^{i,n}, \bar{A}_t^{-i,n})\,dt + g^i(\bar{L}_T^n, \bar{A}_T^{i,n}, \bar{A}_T^{-i,n})\right] = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T h^i(\bar{L}_t, \bar{A}_t^i, \bar{A}_t^{-i})\,dt + g^i(\bar{L}_T, \bar{A}_T^i, \bar{A}_T^{-i})\right]. \tag{4.32}$$

Moreover, Condition (4.4) yields we find

$$\sup_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\sup_{t\in[0,T]}|\bar{f}_t^n|^{\alpha p} + \sup_{t\in[0,T]}|\bar{f}_t|^{\alpha p}\right] = 2\,\mathbb{E}^{\mathbb{P}_0}\left[\sup_{t\in[0,T]}|(\pi_f)_t|^{\alpha p}\right] < \infty. \tag{4.33}$$

The latter, together with (4.10) and (4.19) allows to use Lemma B.3 in Appendix B in order to deduce that

$$\lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T \bar{f}_t^{i,n}\,d\bar{A}_t^{i,n}\right] = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_{[0,T]} \bar{f}_t^i\,d\bar{A}_t^i\right],$$

which, together with (4.32), gives

$$\lim_n \mathcal{J}_{\bar{\beta}}^i(\mathbf{A}^n) = \mathcal{J}_{\bar{\beta}}^i(\bar{A}^i, \bar{A}^{-i}). \tag{4.34}$$

Fix now $M \in \mathbb{N}$ and define the sequence of processes $\{\bar{B}^{n,M}\}_{n\in\mathbb{N}}$ by $\bar{B}_t^{n,M} := \bar{B}_t^n \wedge M$ as well as the process $\bar{B}_t^M := \bar{B}_t \wedge M$. Observe that, for each $n \in \mathbb{N}$, from (4.24) and the definition of $\bar{B}^{n,M}$ we have

$$\mathcal{J}_{\bar{\beta}}^i(\bar{\mathbf{A}}^n) > \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T h^i(\bar{L}_t^n, \bar{B}_t^n, \bar{A}_t^{-i,n})\,dt + g^i(\bar{L}_T^n, \bar{B}_T^n, \bar{A}_T^{-i,n})\right] + \mathbb{E}^{\bar{\mathbb{Q}}}\Big[\int_{[0,T]} \bar{f}_t^{i,n}\,d\bar{B}_t^{n,M}\Big] + \varepsilon. \tag{4.35}$$

Moreover, notice that the convergence established in (4.27) and in (4.28) implies that, $\bar{\mathbb{Q}}$-a.s., the sequence $\{\bar{B}^{n,M}\}_{n\in\mathbb{N}}$ converges to $\bar{B}^M$ in the measure $dt + \delta_T$ on $[0,T]$.

Now, since the sequence $\{\bar{B}^{n,M}\}_{n\in\mathbb{N}}$ is bounded by the constant M, we can use again Lemma B.3 in Appendix B to deduce that

$$\lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\Big[\int_{[0,T]} \bar{f}_t^{i,n}\,d\bar{B}_t^{n,M}\Big] = \mathbb{E}^{\bar{\mathbb{Q}}}\Big[\int_{[0,T]} \bar{f}_t^i\,d\bar{B}_t^M\Big]. \tag{4.36}$$

Hence, thanks to (4.34), (4.31) and (4.36), for each fixed M we can pass to the limit in the inequality (4.35), in order to obtain that

$$\mathcal{J}^i_{\bar{\beta}}(\bar{\mathbf{A}}) \geq \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T h^i(\bar{L}_t, \bar{B}_t, \bar{A}^{-i}_t)\, dt + g^i(\bar{L}_T, \bar{B}_T, \bar{A}^{-i}_T)\right] + \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_{[0,T]} \bar{f}^i_t \, d\bar{B}^M_t\right] + \varepsilon.$$

Finally, by the monotone convergence theorem, we can take the limit as $M \to \infty$ in the latter inequality to deduce that

$$\mathcal{J}^i_{\bar{\beta}}(\bar{A}^i, \bar{A}^{-i}) \geq \mathcal{J}^i_{\bar{\beta}}(\bar{B}, \bar{A}^{-i}) + \varepsilon. \tag{4.37}$$

On the other hand, the probability measure $\bar{\mathbb{Q}} \circ (\bar{f}, \bar{L}, \bar{\mathbf{A}})^{-1}$ is an accumulation point of the sequence $\mathbb{P} \circ (f, L, \mathbf{A}^n)^{-1}$, and hence, by Theorem 4.4, the couple $(\bar{\beta}, \bar{\mathbf{A}})$ is a weak Nash equilibrium of the monotone-follower game, with $\bar{\beta} := (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{Q}}, \bar{f}, \bar{L})$. Moreover, $\bar{B}$ is an admissible strategy associated to the basis $\bar{\beta}$; this implies that

$$\mathcal{J}^i_{\bar{\beta}}(\bar{A}^i, \bar{A}^{-i}) \leq \mathcal{J}^i_{\bar{\beta}}(\bar{B}, \bar{A}^{-i}),$$

which, together with (4.37), leads to a contradiction, and thus completes the proof. □

Remark 4.6. *Theorem 4.5 can also be understood in a different way. Fix a weak Nash equilibrium $(\bar{\beta}, \bar{\mathbf{A}})$ which is an accumulation point of a sequence of Nash equilibria of the n-Lipschitz game on a fixed basis β, and define*

$$\mathcal{V} = (\mathcal{V}^1, ..., \mathcal{V}^N) := (\mathcal{J}^1_{\bar{\beta}}(\bar{\mathbf{A}}), ..., \mathcal{J}^N_{\bar{\beta}}(\bar{\mathbf{A}})).$$

Then, $\mathcal{V}$ is a Nash equilibrium payoff *of the monotone-follower game (see, e.g., Definition 2.7 in [17], or [48]), in the sense that, for each $\varepsilon > 0$, there exists $\mathbf{A}^\varepsilon \in \mathcal{A}^N_\beta$ such that, for each $i = 1, ..., N$, we have:*

(1) $\mathcal{J}^i_\beta(A^{i,\varepsilon}, A^{-i,\varepsilon}) \leq \mathcal{J}^i_\beta(B^i, A^{-i,\varepsilon}) + \varepsilon \quad$ for each $\quad B^i \in \mathcal{A}_\beta$;

(2) $|\mathcal{J}^i_\beta(\mathbf{A}^\varepsilon) - \mathcal{V}^i| \leq \varepsilon$.

Moreover, Theorem 4.5 shows that the Nash equilibrium payoff $\mathcal{V}$ is such that, for each $\varepsilon > 0$, the profile strategy $\mathbf{A}^\varepsilon$, which satisfies the conditions of the definition above, can be chosen as a Nash equilibrium of the n-Lipschitz game, for n large enough.

Remark 4.7. *Notice that the submodularity conditions (2) and (3) in Assumption 2.2 are not necessarily needed in the proof of Theorem 4.4 and 4.5. Indeed, oly the requirement that, for each $n \in \mathbb{N}$, there exists a Nash equilibrium for the n-Lipschitz game is needed. The latter games can be seen as stochastic differential games, where the set of strategies is the set of progressively measurable stochastic processes $u^i : \Omega \times [0, T] \to [0, n]^d$, with degenerate dynamics $A^i_t = \int_0^t u^i_s ds$. This fact suggests that, whenever the submodularity requirement does not hold, one might exploit, on a case by case basis, existence results on equilibria for sochastic differential games (see, e.g., [22] and references therein for results on stochastic differential games).*

5. Applications and Examples

5.1. **Existence of Equilibria in a Class of Stochastic Differential Games.** This subsection is devoted to show that Theorem 2.4 applies to deduce existence of open loop Nash equilibria in stochastic differential games with singular controls, whenever a certain structure is preserved by the dynamics. For the sake of illustration, we propose the following model.

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions and consider on it N standard $\mathbb{F}$-Brownian motions W^i. Suppose to be given, for $i = 1, ..., N$, measurable functions $g^i, h^i : \mathbb{R}^k \times \mathbb{R}^N \to \mathbb{R}$, as well as constants $\mu^i, \sigma^i \in \mathbb{R}$ and continuous $\mathbb{F}$-adapted stochastic processes $f^i : \Omega \times [0, T] \to [0, \infty)$. Assume moreover to be given an $\mathbb{F}$-adapted process $L : [0, T] \times \Omega \to \mathbb{R}^k$ with càdlàg components. The set of admissible strategies $\mathcal{A}$

is defined as the set of nondecreasing, nonnegative, càdlàg, $\mathbb{F}$-adapted, $\mathbb{R}$-valued stochastic processes, whereas $\mathcal{A}^N := \bigotimes_{i=1}^N \mathcal{A}$ denotes the set of asmissible profile strategies.

We consider the N-player stochastic differential game of singular controls in which, for $i = 1, ..., N$, player i chooses an admissible strategy $\xi^i \in \mathcal{A}$ to control her private state, which evolves according to the stochastic differential equation

$$dX_t^i = \mu^i X_t^i \, dt + \sigma^i X_t^i \, dW_t^i + d\xi_t^i, \quad t \in [0,T], \quad X_0^i = x^i > 0, \tag{5.1}$$

in order to minimize her expected cost

$$\mathcal{J}^i(\xi^i, \xi^{-i}) := \mathbb{E}\left[\int_0^T h^i(L_t, X_t^i, X_t^{-i})dt + g^i(L_T, X_T^i, X_T^{-i}) + \int_{[0,T]} f_t^i d\xi_t^i\right].$$

Observe that, for $i = 1, ..., N$, the solution to equation (5.1) is given by

$$X_t^i = E_t^i \left[x^i + \int_{[0,t]} \frac{1}{E_s^i} d\xi_s^i\right] = E_t^i \left[x^i + \bar{\xi}_t^i\right], \tag{5.2}$$

where the processes $\{E_t^i\}_{t\in[0,T]}$ and $\{\bar{\xi}_t^i\}_{t\in[0,T]}$ are defined by

$$E_t^i := \exp\left[\left(\mu^i - \frac{(\sigma^i)^2}{2}\right) t + \sigma^i W_t^i\right] \quad \text{and} \quad \bar{\xi}_t^i := \int_{[0,t]} \frac{1}{E_s^i} d\xi_s^i. \tag{5.3}$$

Assumption 5.1. *Let h^i and g^i satisfy Assumption 2.2. Suppose moreover that:*

(1) *for each $i = 1, ..., N$, there exist functions $\widetilde{H}^i, \widetilde{G}^i : \mathbb{R}^k \times \mathbb{R} \to [0, \infty)$ such that*

$$h^i(l, x^i, x^{-i}) \leq \widetilde{H}^i(l, x^i) \quad \text{and} \quad g^i(l, x^i, x^{-i}) \leq \widetilde{G}^i(l, x^i), \quad \text{for each} \quad (l, x) \in \mathbb{R}^k \times \mathbb{R}^N,$$

with

$$\mathbb{E}\left[\int_0^T \widetilde{H}^i(L_t, E_t^i)\, dt + \widetilde{G}^i(L_T, E_T^i)\right] < \infty;$$

(2) *there exists a constant k_1 such that, for each $i = 1, ..., N$, we have $g^i(l, x) \geq k_1 x^i$ for each $(l, x) \in \mathbb{R}^k \times \mathbb{R}^N$.*

Theorem 5.2. *Under Assumption 5.1, there exists an open-loop Nash equilibrium of the previously introduced stochastic differential game.*

Proof. Thanks to (5.2), the cost functional of player i can be rewritten in terms of $\bar{\xi}^i$ (cf. (5.3)), that is

$$\begin{aligned}\mathcal{J}^i(\xi^i, \xi^{-i}) = \mathbb{E}\bigg[\int_0^T & h^i\left(L_t, E_t^i\left[x^i + \bar{\xi}_t^i\right], \left\{E_t^j\left[x^j + \bar{\xi}_t^j\right]\right\}_{j\neq i}\right)dt \\ & + g^i\left(L_T, E_T^i\left[x^i + \bar{\xi}_T^i\right], \left\{E_T^j\left[x^j + \bar{\xi}_T^j\right]\right\}_{j\neq i}\right) + \int_{[0,T]} f_t^i E_t^i \, d\bar{\xi}_t^i\bigg].\end{aligned} \tag{5.4}$$

This leads to define the new functions $\bar{h}^i, \bar{g}^i : \mathbb{R}^k \times (0,\infty)^N \times \mathbb{R}^N \to [0, \infty)$ by

$$\begin{aligned}\bar{h}^i(l, e, z^i, z^{-i}) &:= h^i(l, e^i[x^i + z^i], \{e^j[x^j + z^j]\}_{j\neq i}) \\ \bar{g}^i(l, e, z^i, z^{-i}) &:= g^i(l, e^i[x^i + z^i], \{e^j[x^j + z^j]\}_{j\neq i}),\end{aligned}$$

as well as the continuous processes $\bar{f}^i : \Omega \times [0,T] \to \mathbb{R}$ by $\bar{f}_t^i := f_t^i E_t^i$. These definitions allows us to introduce new cost functionals in terms of new profile strategies $\zeta = (\zeta^1, ..., \zeta^N) \in \mathcal{A}^N$ setting

$$\bar{\mathcal{J}}^i(\zeta^i, \zeta^{-i}) := \mathbb{E}\bigg[\int_0^T \bar{h}^i(L_t, E_t, \zeta_t^i, \zeta_t^{-i})dt + \bar{g}^i(L_T, E_T, \zeta_T^i, \zeta_T^{-i}) + \int_{[0,T]} \bar{f}_t^i d\zeta_t^i\bigg].$$

Notice that, by (5.4) and the definition of $\bar{\xi}^i$ in (5.3) as a function of ξ^i, we have that

$$\bar{\mathcal{J}}^i(\bar{\xi}^i, \bar{\xi}^{-i}) = \mathcal{J}^i(\xi^i, \xi^{-i}), \quad \forall \, \xi \in \mathcal{A}^N, \quad \forall \, i \in \{1, ..., N\}.$$

Furthermore, for each $\zeta \in \mathcal{A}^N$ there exists a unique $\xi \in \mathcal{A}^N$ such that $\zeta^i = \bar{\xi}^i$ for each $i \in \{1, ..., N\}$. This means that solving the stochastic differential game in the class of profile strategies $\xi \in \mathcal{A}$ and with cost functionals $\mathcal{J}^i$ is equivalent to solve the monotone-follower game for $\zeta \in \mathcal{A}$ and cost functionals $\bar{\mathcal{J}}^i$. The rest of the proof is then mainly devoted to show that the costs $\bar{h}^i$ and $\bar{g}^i$, together with the processes $\bar{f}^i$, satisfy the conditions of Theorem 2.4.

Since the functions h^i and g^i satisfy Assumption 2.2, for each $(l, e, z^{-i}) \in \mathbb{R}^k \times (0, \infty)^N \times \mathbb{R}^{N-1}$ the functions $\bar{h}^i(l, e, \cdot, z^{-i})$ and $\bar{g}^i(l, e, \cdot, z^{-i})$ are clearly continuous and strictly convex. Moreover, for $(l, e) \in \mathbb{R}^k \times (0, \infty)^N$ and $z, \bar{z} \in \mathbb{R}^N$ such that $z \leq \bar{z}$, we have $e^j[x^j + z^j] \leq e^j[x^j + \bar{z}^j]$ for each $j = 1, ..., N$, since the components of e are positive. Therefore, because h^i has decreasing differences, we deduce that

$$\begin{aligned}
&\bar{h}^i(l, e, \bar{z}^i, z^{-i}) - \bar{h}^i(l, e, z^i, z^{-i}) \\
&\qquad = h^i(l, e^i[x^i + \bar{z}^i], \{e^j[x^j + z^j]\}_{j \neq i}) - h^i l, (e^i[x^i + z^i], \{e^j[x^j + z^j]\}_{j \neq i}) \\
&\qquad \geq h^i(l, e^i[x^i + \bar{z}^i], \{e^j[x^j + \bar{z}^j]\}_{j \neq i}) - h^i(l, e^i[x^i + z^i], \{e^j[x^j + \bar{z}^j]\}_{j \neq i}) \\
&\qquad = \bar{h}^i(l, e, \bar{z}^i, \bar{z}^{-i}) - \bar{h}^i(l, e, z^i, \bar{z}^{-i}),
\end{aligned}$$

which means that $\bar{h}^i$ has decreasing difference as well. In the same way it is possible to show that $\bar{g}^i$ has decreasing differences, and this allows to conclude that the functions $\bar{h}^i$ and $\bar{g}^i$ satisfy Assumption 2.2. Moreover, thanks to (1) in Assumption 5.1, Condition 2.6 is clearly satisfied with $r^i(\zeta) = 0$ for each $\zeta \in \mathcal{A}^N$.

We prove now that the functionals $\bar{\mathcal{J}}^i$ satisfy a slightly different version of Condition 2.5. The superlinear condition (2) in Assumption 5.1 implies that

$$\begin{aligned}
\bar{J}^i(\zeta^i, \zeta^{-i}) \geq & \mathbb{E}\left[\bar{g}^i(L_T, \zeta_T^i, \zeta_T^{-i})\right] = \mathbb{E}\left[g^i\left(L_T, E_T^i\left[x^i + \zeta_T^i\right], \left\{E_T^j\left[x^j + \zeta_T^j\right]\right\}_{j \neq i}\right)\right] \\
& \geq k_1 \mathbb{E}\left[E_T^i\left[x^i + \zeta_T^i\right]\right] \geq k_1 \mathbb{E}\left[E_T^i \, \zeta_T^i\right] = k_1 \, \mathbb{E}[E_T^i] \, \mathbb{E}^{\tilde{\mathbb{P}}^i}\left[\zeta_T^i\right],
\end{aligned}$$

where $\tilde{\mathbb{P}}^i$ is the probability measure on $(\Omega, \mathcal{F})$ given by

$$d\tilde{\mathbb{P}}^i := \frac{E_T^i}{\mathbb{E}[E_T^i]} \, d\mathbb{P},$$

and equivalent to $\mathbb{P}$.

We can therefore apply Theorem 2.4 (in fact a slightly different version of it, in which the expectation in Condition 2.5 is replaced by the expectation under an equivalent probability measure) to deduce existence of a Nash equilibrium $\hat{\zeta} = (\hat{\zeta}^1, ..., \hat{\zeta}^N)$ of the monotone-follower game with cost functionals $\bar{\mathcal{J}}^i$. Hence the process $\hat{\xi} = (\hat{\xi}^1, ..., \hat{\xi}^N)$ defined by

$$\hat{\xi}_t^i := \int_{[0,t]} E_s^i \, d\hat{\zeta}_s^i$$

is an open loop Nash equilibrium of the stochastic differential game. □

Remark 5.3. *The same arguments employed in the proof of Theorem 5.2 apply if we replace the dynamics of the controlled geometric Brownian motion in (5.1) by the dynamics of a controlled Ornstein–Uhlenbeck process*

$$dX_t^i = \theta^i(\mu^i - X_t^i) \, dt + \sigma^i \, dW_t^i + d\xi_t^i, \quad t \in [0, T], \quad X_0^i = x^i > 0,$$

for some parameters $\theta^i, \sigma^i > 0$ and $\mu^i \in \mathbb{R}$. Mean-reverting dynamics (as the Ornstein–Uhlenbeck one) find important application in the energy and commodity markets (see, e.g., [12] or Chapter 2 in [50]).

5.2. **An Algorithm to Approximate the Least Nash Equilibrium.** In this subsection we prove that, also in our setting, the algorithm introduced by Topkis (see Algorithm II in [67]) for submodular games converges to the least Nash equilibrium of the game.

According to the notation of Section 2, define the sequence of processes $\{\mathbf{R}^n\}_{n\in\mathbb{N}} \subset \mathcal{A}^N$ in the following way:

- $\mathbf{R}^0 = 0 \in \mathcal{A}^N$;
- for each $n \geq 1$, set $\mathbf{R}^{n+1} := \mathbf{R}(\mathbf{R}^n)$.

Theorem 5.4. *Suppose that the assumptions of Theorem 2.4 hold. Assume, moreover, that there exists a constant $C > 0$ such that, for each $i = 1, ..., N$,*

$$h^i(l,a) + g^i(l,a) \leq C(1+|a|), \quad \forall\,(l,a) \in \mathbb{R}^k \times \mathbb{R}^{Nd} \quad \text{and} \quad |f^i_t| \leq C, \quad \forall\, t \in [0,T], \quad \mathbb{P}-a.s. \tag{5.5}$$

Then the sequence $\{\mathbf{R}^n\}_{n\in\mathbb{N}}$ is monotone increasing in the lattice $(\mathcal{A}^N, \preccurlyeq^N)$ and it converges to the least Nash equilibrium of the game.

Proof. Since the map $\mathbf{R} : \mathcal{A}^N \to \mathcal{A}^N$ is increasing (cf. *Step 3* in the proof of Theorem 2.4), the sequence $\{\mathbf{R}^n\}_{n\in\mathbb{N}}$ is clearly monotone increasing with respect to the order relation in $\mathcal{A}^N$.

Define now the process $\mathbf{S} := (S^1, ..., S^N) \in \mathcal{A}^N$ as the least upper bound of the sequence $\{\mathbf{R}^n\}_{n\in\mathbb{N}}$ in the lattice $(\mathcal{A}^N, \preccurlyeq^N)$. We claim, and prove later, that there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that, for each $\omega \in \Omega \setminus \mathcal{N}$, there exists a countable subset $\mathcal{I}(\omega)$ of $[0,T)$ such that,

$$\mathbf{S}_t(\omega) = \lim_n \mathbf{R}^n_t(\omega) \quad \forall\, t \in [0,T] \setminus \mathcal{I}(\omega), \quad \forall\, \omega \in \Omega \setminus \mathcal{N}. \tag{5.6}$$

We next show that the limit point $\mathbf{S}$ is a Nash equilibrium. By *Step* 2 in the proof of Theorem 2.4, we know that there exists a suitable constant $\widetilde{C}$ such that, for each $n \in \mathbb{N}$, $\mathbb{E}[|\mathbf{R}^n_T|] \leq \widetilde{C}$. Hence, by the monotone convergence theorem, we deduce that

$$\mathbb{E}[|\mathbf{S}_T|] \leq \widetilde{C}. \tag{5.7}$$

Fix then $i \in \{1, ..., N\}$ and $B^i \in \mathcal{A}$. If $\mathbb{E}[|B^i_T|] = \infty$, then, by the coercivity condition (2.5), we would automatically have $\mathcal{J}^i(S^i, S^{-i}) \leq \mathcal{J}^i(B^i, S^{-i}) = \infty$. Hence, without loss of generality, we can assume that

$$\mathbb{E}[|B^i_T|] < \infty. \tag{5.8}$$

Now, since $R^{i,n+1}$ minimizes $\mathcal{J}^i(\cdot, R^{-i,n})$, for each $n \in \mathbb{N}$ we can write

$$\begin{aligned} &\mathbb{E}\bigg[\int_0^T h^i(L_t, R^{i,n+1}_t, R^{-i,n}_t)\,dt + g^i(L_T, R^{i,n+1}_T, R^{-i,n}_T) + \int_{[0,T]} f^i_t\, dR^{i,n+1}_t\bigg] \\ &\quad \leq \mathbb{E}\bigg[\int_0^T h^i(L_t, B^i_t, R^{-i,n}_t)\,dt + g^i(L_T, B^i_T, R^{-i,n}_T) + \int_{[0,T]} f^i_t\, dB^i_t\bigg]. \end{aligned}$$

Moreover, the limit in (5.6), together with conditions (5.5) and the estimates (5.7) and (5.8), allows us to invoke the dominated convergence theorem and to take the limit as n goes to infinity in the last inequality in order to deduce that $\mathcal{J}^i(S^i, S^{-i}) \leq \mathcal{J}^i(B^i, S^{-i})$. Hence $\mathbf{S}$ is a Nash equilibrium.

Finally, we prove that $\mathbf{S}$ is the least Nash equilibrium. Suppose that $\bar{\mathbf{S}}$ is another Nash equilibrium. By definition we have $\mathbf{R}^0 = 0 \preccurlyeq^N \bar{\mathbf{S}}$. If, for an arbitrary $n \in \mathbb{N}$, we have $\mathbf{R}^n \preccurlyeq^N \bar{\mathbf{S}}$, then, since the map $\mathbf{R}$ is increasing and $\bar{\mathbf{S}}$ is a fixed point of $\mathbf{R}$, we have $\mathbf{R}^{n+1} = \mathbf{R}(\mathbf{R}^n) \preccurlyeq^N \mathbf{R}(\bar{\mathbf{S}}) = \bar{\mathbf{S}}$. Hence, by induction, we deduce that $\mathbf{R}^n \preccurlyeq^N \bar{\mathbf{S}}$ for each $n \in \mathbb{N}$, which in turn implies that $\mathbf{S} \preccurlyeq^N \bar{\mathbf{S}}$, since $\mathbf{S}$ is the least upper bound of the sequence $\{\mathbf{R}^n\}_{n\in\mathbb{N}}$.

To conclude the proof it only remains to prove the limit in (5.6). Recall the construction of $\mathbf{S}$ and $\tilde{\mathbf{S}}$ (cf. (2.9) and (2.10) in *Step 1* in the proof of Theorem 2.4). Notice that, since the sequence $\{\mathbf{R}^n\}_{n\in\mathbb{N}}$ is increasing in the lattice $(\mathcal{A}^N, \preccurlyeq^N)$, there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that

$$\mathbf{R}^n_t(\omega) \leq \mathbf{R}^{n+1}_t(\omega), \quad \forall\, t \in [0,T], \quad \forall\, n \in \mathbb{N}, \quad \forall\, \omega \in \Omega \setminus \mathcal{N}. \tag{5.9}$$

Hence we can assume

$$\tilde{\mathbf{S}}_q(\omega) = \lim_n \mathbf{R}^n_q(\omega), \quad \forall\, q \in Q := ([0,T] \cap \mathbb{Q}) \cup \{T\}, \quad \forall\, \omega \in \Omega \setminus \mathcal{N}.$$

Since, for each $\omega \in \Omega \setminus \mathcal{N}$, $\mathbf{S}_T(\omega) = \tilde{\mathbf{S}}_T(\omega)$ by definition, then the limit in (5.6) is verified in T. Take now $\bar{t} \in (0,T)$. If, for $\omega \in \Omega \setminus \mathcal{N}$, the limit in (5.6) does not hold in $\bar{t}$, then there exists $\varepsilon > 0$ such that

$$\sup_n \mathbf{R}^n_{\bar{t}}(\omega) + \varepsilon \leq \mathbf{S}_{\bar{t}}(\omega).$$

Hence, for $q \in Q$ such that $q < \bar{t}$, we find

$$\tilde{\mathbf{S}}_q(\omega) + \varepsilon = \sup_n \mathbf{R}^n_q(\omega) + \varepsilon \leq \sup_n \mathbf{R}^n_{\bar{t}}(\omega) + \varepsilon \leq \mathbf{S}_{\bar{t}}(\omega).$$

This implies that, whenever $s < \bar{t}$, we have

$$\mathbf{S}_s(\omega) + \varepsilon := \inf_{s<q<\bar{t},\, q\in Q} \tilde{\mathbf{S}}_q(\omega) + \varepsilon \leq \mathbf{S}_{\bar{t}}(\omega),$$

and hence that

$$\mathbf{S}_{\bar{t}-}(\omega) + \varepsilon := \sup_{s<\bar{t}} \mathbf{S}_s(\omega) + \varepsilon \leq \mathbf{S}_{\bar{t}}(\omega).$$

The latter means that $\bar{t}$ is in the set $\mathcal{I}(\omega)$ of discontinuity points of $\mathbf{S}(\omega)$. Because $\mathcal{I}(\omega)$ is countable, the limit in (5.6) is then proved, and the proof is completed. $\square$

6. An Extension of Theorem 2.4 with Regular Controls

In this section we generalize Theorem 2.4 to the case in which players are allowed to choose both a regular and a singular control.

Recall the notation introduced in Section 2. Fix a square-integrable random variable Θ and define the space of *regular controls*

$$\mathcal{U} := \left\{ u : \Omega \times [0,T] \to \mathbb{R}^d \,\middle|\, u \text{ is } \bar{\mathbb{F}}^{f,L}_+\text{-progressively measurable and } |u_t| \leq \Theta \ \mathbb{P} \otimes dt - \text{a.e.} \right\},$$

the space of admissible strategies $\mathcal{U} \times \mathcal{A}$, and the space of admissible profile strategies $(\mathcal{U} \times \mathcal{A})^N := \bigotimes_{i=1}^N \mathcal{U} \times \mathcal{A}$. Elements of $\mathcal{U} \times \mathcal{A}$ will be denoted by $X = (u, A)$, while elements of $(\mathcal{U} \times \mathcal{A})^N$ will be denoted by $\mathbf{X} = (X^1, ..., X^N)$.

For each $i = 1, ..., N$, consider measurable functions $h^i, g^i : \mathbb{R}^k \times \mathbb{R}^{2Nd} \to [0, \infty)$. We define the game in which each player $i \in \{1, ..., N\}$ is allowed to choose an admissible strategy $X^i = (u^i, A^i) \in \mathcal{U} \times \mathcal{A}$ in order to minimize the cost functional

$$\mathcal{J}^i(X^i, X^{-i}) := \mathbb{E}\left[\int_0^T h^i(L_t, \mathbf{X}_t)\, dt + g^i(L_T, \mathbf{A}_T) + \int_{[0,T]} f^i_t \, dA^i_t \right], \tag{6.1}$$

where $X^{-i} := (X^j)_{j \neq i}$ and $\mathbf{X} := (X^i, X^{-i}) \in (\mathcal{U} \times \mathcal{A})^N$.

Next, on the space of admissible strategies $\mathcal{U} \times \mathcal{A}$, we define the order relation $\preccurlyeq$ such that, for $X, Y \in \mathcal{U} \times \mathcal{A}$, one has

$$X \preccurlyeq Y \quad \Longleftrightarrow \quad u_t \leq v_t \quad \text{and} \quad A_t \leq B_t \quad \mathbb{P} \otimes dt - \text{a.e.}$$

Moreover, endow the space $\mathcal{U} \times \mathcal{A}$ with a lattice structure, defining the processes $X \wedge Y := (u \wedge v, A \wedge B)$ and $X \vee Y := (u \vee v, A \vee B)$ where

$$(u \wedge v)_t := u_t \wedge v_t \quad \text{and} \quad (u \vee v)_t := u_t \vee v_t \quad \mathbb{P} \otimes dt - \text{a.e.},$$

and

$$(A \wedge B)_t := A_t \wedge B_t \quad \text{and} \quad (A \vee B)_t := A_t \vee B_t \quad \mathbb{P} \otimes dt - \text{a.e.}$$

In the same way, on the set of profile strategies $(\mathcal{U} \times \mathcal{A})^N$, define, for $\mathbf{X}, \mathbf{Y} \in (\mathcal{U} \times \mathcal{A})^N$, an order relation $\preccurlyeq^N$ by

$$\mathbf{X} \preccurlyeq^N \mathbf{Y} \quad \iff \quad X^i \preccurlyeq Y^i \quad \forall\, i \in \{1, ..., N\},$$

together with the lattice structure

$$\mathbf{X} \wedge \mathbf{Y} := (X^1 \wedge Y^1, ..., X^N \wedge Y^N) \quad \text{and} \quad \mathbf{X} \vee \mathbf{Y} := (X^1 \vee Y^1, ..., X^N \vee Y^N).$$

Then, Theorem 2.4 admits the following generalization.

Theorem 6.1. *For each $i = 1, ..., N$, assume that:*

(1) for each $(l, x^{-i}) \in \mathbb{R}^k \times \mathbb{R}^{2(N-1)d}$, the functions $h^i(l, \cdot, x^{-i})$ and $g^i(l, \cdot, x^{-i})$ are lower semi-continuous, and strictly convex;

(2) for each $l \in \mathbb{R}^k$ the functions $h^i(l, \cdot, \cdot)$ and $g^i(l, \cdot, \cdot)$ have decreasing differences in (x^i, x^{-i});

(3) for each $(l, x^{-i}) \in \mathbb{R}^k \times \mathbb{R}^{2(N-1)d}$, the functions $h^i(l, \cdot, x^{-i})$ and $g^i(l, \cdot, x^{-i})$ are submodular;

(4) there exist two constants $K, \kappa > 0$ such that

$$\mathcal{J}^i(X^i, X^{-i}) \geq \kappa\, \mathbb{E}\left[|A^i_T|\right]$$

for all $\mathbf{X} \in (\mathcal{U} \times \mathcal{A})^N$ with

$$\mathbb{E}\left[|A^i_T|\right] \geq K;$$

(5) there exists a constant $M > 0$ such that

$$\text{for all} \quad \mathbf{X} \in (\mathcal{U} \times \mathcal{A})^N \quad \text{there exists} \quad r^i(\mathbf{X}) \in \mathcal{U} \times \mathcal{A} \quad \text{such that} \quad \mathcal{J}^i(r^i(\mathbf{X}), X^{-i}) \leq M.$$

Then the set of Nash equilibria $F \subset (\mathcal{U} \times \mathcal{A})^N$ is non empty, and the partially ordered set $(F, \preccurlyeq^N)$ is a complete lattice.

Proof. For $w := \frac{2M}{\kappa} \vee K$, recall the definition of $\mathcal{A}(w)$ given in (2.7). Combining arguments from *Step 2* of the proof of Theorem 2.4, and from *Step 2* of the proof of Theorem 3.1, it is possible to show that the best reply maps $R^i : (\mathcal{U} \times \mathcal{A})^N \to \mathcal{U} \times \mathcal{A}(w)$ are well defined. Moreover, the same reasoning employed in *Step 3* of the proof of Theorem 2.4 allows us to deduce that the best reply maps are increasing with respect to the order relations on $(\mathcal{U} \times \mathcal{A})^N$ and $\mathcal{U} \times \mathcal{A}$. Then, in order to complete the proof, it remains to show that the lattice $((\mathcal{U} \times \mathcal{A}(w))^N, \preccurlyeq^N)$ is complete, and, in view of *Step 1* of the proof of Theorem 2.4, it is enough to prove that the lattice $(\mathcal{U}, \preccurlyeq)$ is complete (where, by a slight abuse of notation, we indicate by $\preccurlyeq$ the order relation on $\mathcal{U}$).

Define on the lattice $(\mathcal{U}, \preccurlyeq)$ the topology $\mathcal{I}$ of intervals (see, e.g., p. 250 in [14]); that is, the topology for which the topology of closed sets is generated by the family of sets $\mathcal{I}_z := \{u \in \mathcal{U} \,:\, u \preccurlyeq z\}$ and $\mathcal{I}^z := \{u \in \mathcal{U} \,:\, z \preccurlyeq u\}$ for $z \in \mathcal{U}$.

We now aim at proving that the topology $\mathcal{I}$ is included in the weak topology of $\mathbb{L}^2([0,T] \times \Omega; \mathbb{R}^d)$. To accomplish that, we show that, for each $z \in \mathcal{U}$, the set $\mathcal{I}_z$ is closed for the weak topology σ of $\mathbb{L}^2([0,T] \times \Omega; \mathbb{R}^d)$ by proving that $\mathcal{I}_z$ is convex and closed for the strong topology of $\mathbb{L}^2([0,T] \times \Omega; \mathbb{R}^d)$.

Take $\{u^n\}_{n \in \mathbb{N}} \subset \mathcal{I}_z$ converging to u in $\mathbb{L}^2([0,T] \times \Omega; \mathbb{R}^d)$. Since, for each $n \in \mathbb{N}$, $u^n \in \mathcal{U}$, we clearly have that $u \in \mathcal{U}$. Furthermore, by definition of $\mathcal{I}_z$, we have, $u^n_t - z_t \leq 0$ $\mathbb{P} \otimes dt$-a.e. This implies that, $\mathbb{P} \otimes dt$-a.e., we have $(u^n_t - z_t)\mathbb{1}_{\{u_t - z_t > 0\}} \leq 0$, so that

$$\mathbb{E}\left[\int_0^T (u^n_t - z_t)\mathbb{1}_{\{u_t - z_t > 0\}}\, dt\right] \leq 0, \quad \forall\, n \in \mathbb{N}.$$

Now, since $\{u^n\}_{n\in\mathbb{N}}$ converges strongly to u, hence weakly, we deduce that

$$\mathbb{E}\left[\int_0^T (u_t - z_t)\mathbb{1}_{\{u_t - z_t > 0\}}\, dt\right] \leq 0, \quad \forall\, n \in \mathbb{N},$$

which in turns implies that, $u_t - z_t \leq 0$ $\mathbb{P}\otimes dt$-a.e. Therefore, $u \in \mathcal{I}_z$, which proves that $\mathcal{I}_z$ is closed for the strong topology. Since the set $\mathcal{I}_z$ is convex, $\mathcal{I}_z$ is also closed for the weak topology. Analogous arguments show that $\mathcal{I}^z$ is closed for the weak topology, and this allows us to conclude that the topology $\mathcal{I}$ is contained in the weak topology σ.

Now, since the space $\mathcal{U}$ is weakly closed and bounded for the $\mathbb{L}^2([0,T]\times\Omega;\mathbb{R}^d)$ norm, by Alaoglu's theorem (see Theorem 6.21 at p. 235 in [1]) it follows that it is weakly compact, and then, because of the inclusion $\mathcal{I}\subset\sigma$, we deduce that $\mathcal{U}$ is compact for the topology $\mathcal{I}$. By a characterization of complete lattices (see Theorem 20 at p. 250 in [14]), from the compactness of $\mathcal{U}$ in the interval topology $\mathcal{I}$ it follows that the lattice $(\mathcal{U},\preccurlyeq)$ is complete, and this proves the theorem. □

Appendix A. Technical Lemmata

Throughout the rest of this technical appendix we assume that Assumption 4.1 holds.

Proof of Lemma 4.2. Fix $i\in\{1,...,N\}$. By convexity of h^i and g^i (cf. Assumption 4.1) we have

$$\begin{aligned}
&\mathcal{J}^i_\beta(B^i, A^{-i}) - \mathcal{J}^i_\beta(A^i, A^{-i})\\
&= \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T \big(h^i(L_t, B^i_t, A^{-i}_t) - h^i(L_t, A^i_t, A^{-i}_t)\big)\, dt + \int_{[0,T]} f^i_t(dB^i_t - dA^i_t)\bigg]\\
&\quad + \mathbb{E}^{\mathbb{P}}\left[g^i(L_T, B^i_T, A^{-i}_T) - g^i(L_T, A^i_T, A^{-i}_T)\right]\\
&\geq \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T \nabla_i h^i(L_t, A^i_t, A^{-i}_t)(B^i_t - A^i_t)\, dt + \int_{[0,T]} f^i_t(dB^i_t - dA^i_t)\bigg]\\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\nabla_i g^i(L_T, A^i_T, A^{-i}_T)(B^i_T - A^i_T)\right].
\end{aligned}$$

Then, integrating by parts and recalling (4.6), we obtain

$$\begin{aligned}
&\mathcal{J}^i_\beta(B^i, A^{-i}) - \mathcal{J}^i_\beta(A^i, A^{-i})\\
&\geq \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T \left(\int_t^T \nabla_i h^i(L_s, A^i_s, A^{-i}_s)ds\right)(dB^i_t - dA^i_t) + \int_{[0,T]} f^i_t(dB^i_t - dA^i_t)\bigg]\\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \nabla_i h^i(L_s, A^i_s, A^{-i}_s)ds\right)(B^i_0 - A^i_0)\right]\\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\nabla_i g^i(L_T, A^i_T, A^{-i}_T)(B^i_T - A^i_T)\right]\\
&= \mathbb{E}^{\mathbb{P}}\bigg[\int_{[0,T]} \left(\int_t^T \nabla_i h^i(L_s, A^i_s, A^{-i}_s)ds\right)(dB^i_t - dA^i_t) + \int_{[0,T]} f^i_t(dB^i_t - dA^i_t)\bigg]\\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\nabla_i g^i(L_T, A^i_T, A^{-i}_T)(B^i_T - A^i_T)\right]\\
&= \mathbb{E}^{\mathbb{P}}\bigg[\int_{[0,T]} Y^i_t(dB^i_t - dA^i_t)\bigg].
\end{aligned}$$

□

Proof of Lemma 4.3. Fix $i\in\{1,...,N\}$ and $n\in\mathbb{N}$. Define the $\mathbb{R}^{Nd}$-valued process $u^n = (u^{1,n},...,u^{N,n})$ as the time derivative of the process $\mathbf{A}^n$, i.e., for $i=1,...,N$, set $u^{i,n}_t := \frac{d}{dt}A^{i,n}_t$ $\mathbb{P}\otimes dt$-a.e.

Take $\varepsilon > 0$ and consider an arbitrary admissible n-Lipschitz strategy $B^i \in \mathcal{L}_\beta(n)$. Define then $v_t^i := \frac{d}{dt} B_t^i$ $\mathbb{P} \otimes dt$-a.e., and $A^{i,\varepsilon} := A^{i,n} + \varepsilon\, (B^i - A^{i,n})$. Since u^n is a Nash equilibrium of the n-Lipschitz game, then $A^{i,n}$ minimizes $\mathcal{J}_\beta^i(\cdot, A^{-i,n})$. Hence, by employing Lemma 4.2, the definition of $A^{i,\varepsilon}$ and setting $\mathbb{P}$-a.s.,

$$Y_t^{i,n,\varepsilon} := \int_t^T \nabla_i h^i(L_t, A_t^{i,\varepsilon}, A_t^{-i,n})\, dt + \nabla_i g^i(L_T, A_T^{i,\varepsilon}, A_T^{-i,n}) + f_t^i, \quad t \in [0,T],$$

we have

$$0 \geq \mathbb{E}^{\mathbb{P}}\left[C^i(L, A^{i,n}, A^{-i,n})\right] - \mathbb{E}^{\mathbb{P}}\left[C^i(L, A^{i,\varepsilon}, A^{-i,n})\right] \geq \varepsilon\, \mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t^{i,n,\varepsilon}\, (dA_t^{i,n} - dA_t^{\varepsilon})\right].$$

After dividing by ε, we obtain

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t^{i,n,\varepsilon}\, (u_t^{i,n} - v_t^i)\, dt\right] \leq 0.$$

We claim (and prove later) that by taking limits as $\varepsilon \to 0$ we have

$$\lim_{\varepsilon \to 0} \mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t^{i,n,\varepsilon}\, (u_t^{i,n} - v_t^i)\, dt\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t^{i,n}\, (u_t^{i,n} - v_t^i)\, dt\right], \tag{A.1}$$

and hence

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t^{i,n}\, (u_t^{i,n} - v_t^i)\, dt\right] \leq 0,$$

which implies that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T (Y_t^{i,n})^+\, (u_t^{i,n} - v_t^i)\, dt\right] \leq \mathbb{E}^{\mathbb{P}}\left[\int_0^T (Y_t^{i,n})^-\, (u_t^{i,n} - v_t^i)\, dt\right].$$

Now, for $j = 1, ..., d$, denote by e_j the jth element of the canonical basis of $\mathbb{R}^d$. Taking $v_t^i := n \sum_{j=1}^d \mathbb{1}_{\{Y_t^{i,n,j} \leq 0\}}\, e_j$, we find

$$0 \leq \mathbb{E}^{\mathbb{P}}\left[\int_0^T (Y_t^{i,n})^+\, u_t^{i,n}\, dt\right] \leq \mathbb{E}^{\mathbb{P}}\left[\int_0^T (Y_t^{i,n})^-\, (u_t^{i,n} - n\,\mathbf{1})\, dt\right],$$

and, because the right-hand side of the latter is non positive as $u^{i,n} \leq n$, we have

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t^{i,n}\, u_t^{i,n}\, dt\right] = -n\, \mathbb{E}^{\mathbb{P}}\left[\int_0^T (Y_t^{i,n})^- \mathbf{1}\, dt\right],$$

which proves (4.8) upon recalling that $dA_t^{i,n} = u_t^{i,n}\, dt$ $\mathbb{P}$-a.s.

In order to prove (4.9) notice that Lemma 4.2 and (4.7) give

$$0 \leq \mathbb{E}^{\mathbb{P}}\left[C^i(f, L, \mathbf{A}^n)\right] \leq \mathbb{E}^{\mathbb{P}}\left[C^i(f, L, 0, A^{-i,n})\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t^{i,n} dA_t^{i,n}\right].$$

Hence from (4.8) and the latter we have that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T (Y_t^{i,n})^- \mathbf{1}\, dt\right] \leq \frac{\mathbb{E}^{\mathbb{P}}\left[C^i(f, L, 0, A^{-i,n})\right]}{n} \leq \frac{1}{n}\mathbb{E}^{\mathbb{P}_0}\left[\int_0^T |H^i((\pi_L)_s)|\, ds + |G((\pi_L)_T)|\right],$$

which, thanks to Condition 2 in Assumption 4.1, implies (4.9).

In order to complete the proof it only remains to prove (A.1). To do so, since $|u_t^{i,n} - v_t| \leq 2n$ for all $t \in [0,T]$ $\mathbb{P}$-a.s., it is enough to prove that $Y^{i,n,\varepsilon}$ converges to $Y^{i,n}$ in $\mathbb{L}^1(\Omega \times [0,T])$.

Notice that, for a suitable constant $\widetilde{C} > 0$, we have

(A.2)
$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|Y_t^{i,n,\varepsilon} - Y_t^{i,n}\right| dt\right] \leq \widetilde{C}\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\nabla_i h^i(L_t, A_t^{i,n,\varepsilon}, A_t^{-i,n}) - \nabla_i h^i(L_t, A_t^{i,n}, A_t^{-i,n})\right| dt\right] + \widetilde{C}\mathbb{E}^{\mathbb{P}}\left[\left|\nabla_i g^i(L_T, A_T^{i,n,\varepsilon}, A_T^{-i,n}) - \nabla_i g^i(L_T, A_T^{i,n}, A_T^{-i,n})\right|\right].$$

Moreover, for $\varepsilon \to 0$ we have that $A_t^{i,\varepsilon}$ converges to $A_t^{i,n}$ for each $t \in [0,T]$, $\mathbb{P}$-a.s. Thus, by the continuity of $\nabla_i h^i$ and $\nabla_i g^i$, we have, $\mathbb{P}$-a.s.

$$\lim_{\varepsilon\to 0}\left|\nabla_i h^i(L_t, A_t^{i,n,\varepsilon}, A_t^{-i,n}) - \nabla_i h^i(L_t, A_t^{i,n}, A_t^{-i,n})\right| = 0 \quad \forall t \in [0,T], \tag{A.3}$$

and

$$\lim_{\varepsilon\to 0}\left|\nabla_i g^i(L_T, A_T^{i,n,\varepsilon}, A_T^{-i,n}) - \nabla_i g^i(L_T, A_T^{i,n}, A_T^{-i,n})\right| = 0. \tag{A.4}$$

Furthermore, since we are always considering integrals of processes bounded by n, we have that $|A_T^{-i,n}| \leq nd(N-1)T$ and, for $\varepsilon < 1$, that $|A_T^{i,\varepsilon}| \leq 2ndT$. Hence, thanks to the growth conditions (2) in Assumption 4.1, we find

$$\begin{aligned}
&|\nabla_i h^i(L_t, A_t^{i,\varepsilon}, A_t^{-i,n})|^p \leq \widetilde{C}\Big(1 + \sup_{[0,T]}(|L_s|^{\gamma_1 p})\Big) =: \widetilde{C}\eta \quad \forall t \in [0,T],\\
&|\nabla_i g^i(L_T, A_T^{i,\varepsilon}, A_T^{-i,n})|^p \leq \widetilde{C}\eta,\\
&|\nabla_i h^i(L_t, A_t^{i,n}, A_t^{-i,n})|^p \leq \widetilde{C}\eta \quad \forall t \in [0,T],\\
&|\nabla_i g^i(L_T, A_T^{i,n}, A_T^{-i,n})|^p \leq \widetilde{C}\eta,
\end{aligned}$$

where $p > 1$ is as in the Assumption 4.1.

The previous estimates together with the integrability condition (4.4) in Assumption 4.1 imply that the two members on the right-hand side of (A.2) are uniformly integrable. This fact, together with the limits in (A.3) and (A.4), allows us to invoke the dominated convergence theorem in order to conclude that the two members on the right-hand side of (A.2) converge to 0 as $\varepsilon \to 0$. This completes the proof of the lemma. □

Lemma A.1. *For $q > 1$ as in Assumption 4.1, we have that*

$$\sup_n \mathbb{E}^{\mathbb{P}}\left[|\mathbf{A}_T^n|^q\right] < \infty.$$

Proof. Fix $i \in \{1, ..., N\}$ and $r > 0$, and define the $\mathbb{R}^d$-valued process

$$A_t^{i,n,r} := A_{t\wedge T^{i,n}(r)}^{i,n} \tag{A.5}$$

where $T^{i,n}(r) := \inf\{\, t \in [0,T] \,:\, A_t^{i,n} \geq r \,\}$, with the usual convention that $\inf\varnothing = \infty$. Now, since $A^{i,n,r}$ is not the best response to $A^{-i,n}$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\int_0^T h^i(L_t, A_t^{i,n}, A_t^{-i,n})\, dt + g^i(L_T, A_T^{i,n}, A_T^{-i,n}) + \int_{[0,T]} f_t^i \, dA_t^{i,n}\right]\\
&\quad \leq \mathbb{E}^{\mathbb{P}}\left[\int_0^T h^i(L_t, A_t^{i,n,r}, A_t^{-i,n})\, dt + g^i(L_T, A_T^{i,n,r}, A_T^{-i,n}) + \int_{[0,T]} f_t^i \, dA_t^{i,n,r}\right].
\end{aligned}$$

Therefore, using (A.5),

$$\mathbb{E}^{\mathbb{P}}\Big[\int_{T\wedge T^{i,n}(r)}^{T} f^i_t\, dA^{i,n}_t\Big] \leq \mathbb{E}^{\mathbb{P}}\Big[\int_{T\wedge T^{i,n}(r)}^{T} \Big(h^i(L_t, r, A^{-i,n}_t) - h^i(L_t, A^{i,n}_t, A^{-i,n}_t)\Big)\, dt\Big] \tag{A.6}$$

$$+\, \mathbb{E}^{\mathbb{P}}\Big[g^i(L_T, A^{i,n,r}_T, A^{-i,n}_T) - g^i(L_T, A^{i,n}_T, A^{-i,n}_T)\Big] \tag{A.7}$$

Thanks to the assumption (4.5) in Assumption 4.1, we also find

$$c\,\mathbb{E}^{\mathbb{P}}\Big[(A^{i,n}_T - r)\,\mathbb{1}_{\{A^{i,n}_T > r\}}\Big] \leq \mathbb{E}^{\mathbb{P}}\Big[\int_{T\wedge T^{i,n}(r)}^{T} f^i_t\, dA^{i,n}_t\Big]. \tag{A.8}$$

Moreover, on the event $\{A^{i,n}_T > r\}$, we have that $r \in [0, A^{i,n}_t]$ for each $t \in [T \wedge T^{i,n}(r), T]$. Hence r is a convex combination of 0 and $A^{i,n}_t$ for each $t \in [T\wedge T^{i,n}(r), T]$, and by convexity of h^i and g^i we find

$$\begin{aligned} &\int_{T\wedge T^{i,n}(r)}^{T} h^i(L_t, r, A^{-i,n}_t)\, dt + g^i(L_T, r, A^{-i,n}_T) \\ &\qquad \leq \int_{T\wedge T^{i,n}(r)}^{T} h^i(L_t, A^{i,n}_t, A^{-i,n}_t)\, dt + \int_0^T h^i(L_t, 0, A^{-i,n}_t)\, dt \\ &\qquad\quad + g^i(L_T, A^{i,n}_T, A^{-i,n}_T) + g^i(L_T, 0, A^{-i,n}_T). \end{aligned} \tag{A.9}$$

Thus, by using (A.8) and (A.9) in (A.6), we obtain

$$\begin{aligned} c\,\mathbb{E}^{\mathbb{P}}\Big[(A^{i,n}_T - r)\,\mathbb{1}_{\{A^{i,n}_T > r\}}\Big] &\leq \mathbb{E}^{\mathbb{P}}\Big[\mathbb{1}_{\{A^{i,n}_T > r\}}\Big(\int_0^T h^i(L_t, 0, A^{-i,n}_t)\, dt + g^i(L_T, 0, A^{-i,n}_T)\Big)\Big] \\ &\leq \mathbb{E}^{\mathbb{P}}\Big[\mathbb{1}_{\{A^{i,n}_T > r\}}\Big(\int_0^T |H^i(L_s)|\, ds + |G^i(L_T)|\Big)\Big]. \end{aligned}$$

Thanks to the integrability condition (4.4) in Assumption 4.1, we can apply Lemma 33 in [47] to conclude that

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}}[|A^{i,n}_T|^q] < q\,\widetilde{C}\,\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |Z^i(L_s)|^q\, ds + +|G^i(L_T)|^q\Big] < \infty,$$

for a suitable constant $\widetilde{C} > 0$, which finally implies the claim of the Lemma. □

Lemma A.2. *There exists $\mathcal{G} \in \bar{\mathcal{F}}$ of full $\bar{\mathbb{Q}}$-measure such that for each $\omega \in \mathcal{G}$ there exists a constant $M(\omega) < \infty$ such that*

$$\sup_n \sup_{t\in[0,T]} \big(|\bar{L}^n_t(\omega)| + |\bar{\mathbf{A}}^n_t(\omega)| + |\bar{L}_t(\omega)| + |\bar{\mathbf{A}}_t(\omega)|\big) \leq M(\omega).$$

Proof. Recalling (4.13) and (4.10), by Fatou's lemma we find

$$\mathbb{E}^{\bar{\mathbb{Q}}}[|\bar{\mathbf{A}}_T|^q] \leq \sup_n \mathbb{E}^{\bar{\mathbb{Q}}}[|\bar{\mathbf{A}}^n_T|^q] = \sup_n \mathbb{E}^{\mathbb{P}}[|\mathbf{A}^n_T|^q] < \infty. \tag{A.10}$$

Hence $\bar{\mathbb{Q}}$-a.s., $|\mathbf{A}^n_T| < \infty$, and again by the convergence in (4.13) we deduce that $\bar{\mathbb{Q}}$-a.s.

$$\sup_n \sup_{t\in[0,T]} \bar{\mathbf{A}}^n_t = \sup_n \bar{\mathbf{A}_T}^n < \infty. \tag{A.11}$$

We now show that, $\bar{\mathbb{Q}}$-a.s.,

$$\sup_n \sup_{t\in[0,T]} |\bar{L}^n_t| < \infty.$$

Let $Q := ([0,T)\cap\mathbb{Q}) \cup \{T\}$ and define the measurable function $\Phi : \mathcal{D}^k \to \mathbb{R}$ by

$$\Phi(X) := \sup_{t\in Q} |X_t|.$$

Then, as in the proof of *(Claim 1)* of Theorem 4.4, it is possible to show that the sequence $\{(f, \Phi(L), L, \mathbf{A}^n)\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{C}_+^{Nd} \times \mathbb{R} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$. Indeed, $\Phi(L)$ is constant w.r.t. n and integrable, and hence tight in $\mathcal{P}(\mathbb{R})$. Thus (modulo taking another subsequence) the application of Skorokhod representation theorem reveals that there exist $\mathbb{R}$-valued random variables $\bar{M}^n$ and $\bar{M}$ on $\bar{\Omega}$ such that: $\bar{\mathbb{Q}} \circ (\bar{f}^n, \bar{M}^n, \bar{L}^n, \bar{\mathbf{A}}^n)^{-1} = \mathbb{P} \circ (f, \Phi(L), L, \mathbf{A}^n)^{-1}$, $\bar{\mathbb{Q}} \circ (\bar{f}, \bar{M}, \bar{L}, \bar{\mathbf{A}})^{-1} = \bar{\mathbb{P}}$ and $\bar{M}^n$ converges to $\bar{M}$ $\bar{\mathbb{Q}}$-a.s. Furthermore, $\bar{\mathbb{Q}} \circ (\bar{M}^n, \bar{L}^n)^{-1}$ is constantly equal to $\mathbb{P} \circ (\Phi(L), L)^{-1}$, and then the same holds for the limit; that is $\bar{\mathbb{Q}} \circ (\bar{M}, \bar{L})^{-1} = \mathbb{P} \circ (\Phi(L), L)^{-1}$. This implies that, for each $n \in \mathbb{N}$,

$$\bar{\mathbb{Q}}[|\bar{M}^n - \Phi(\bar{L}^n)| = 0] = \bar{\mathbb{Q}}[|\bar{M} - \Phi(\bar{L})| = 0] = \mathbb{P}[|\Phi(L) - \Phi(L)| = 0] = 1,$$

hence $\bar{M}^n = \Phi(\bar{L}^n)$ and $\bar{M} = \Phi(\bar{L})$ $\bar{\mathbb{Q}}$-a.s.

Now, thanks to the integrability condition (4.4) in Assumption 4.1, we have

$$\mathbb{E}^{\bar{\mathbb{Q}}}[\bar{M}] = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\sup_{t\in[0,T]} |\bar{L}_t|\right] = \mathbb{E}^{\mathbb{P}}[\Phi(L)] = \mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]} |L_t|\right] < \infty. \tag{A.12}$$

Then $\bar{M} < \infty$ $\bar{\mathbb{Q}}$-a.s., and since $\Phi(\bar{L}^n)$ converges to $\bar{M}$ $\bar{\mathbb{Q}}$-a.s., we finally deduce that

$$\sup_n \sup_{t\in[0,\infty)} |\bar{L}_t^n| = \sup_n \Phi(\bar{L}^n) < \infty. \tag{A.13}$$

Combining then (A.10), (A.11), (A.12) and (A.13) we find the thesis. $\square$

Lemma A.3. *For every $i = 1, ..., N$, $\bar{Y}^{i,n}$ converges to $\bar{Y}^i$ uniformly on the interval $[0, T]$, $\bar{\mathbb{Q}}$-a.s.*

Proof. First we prove that $\bar{\mathbb{Q}}$-a.s.

$$\lim_n \sup_{t\in[0,T]} \left| \int_t^T \left(\nabla_i h^i(\bar{L}_s^n, \bar{\mathbf{A}}_s^n) - \nabla_i h^i(\bar{L}_s, \bar{\mathbf{A}}_s)\right) ds \right| = 0. \tag{A.14}$$

From Lemma A.2, there exists $\mathcal{G}$ of full $\bar{\mathbb{Q}}$-measure such that for each $\omega \in \mathcal{G}$ we have

$$\sup_n \sup_{t\in[0,T]} \left(|\bar{L}_t^n(\omega)| + |\bar{\mathbf{A}}_t^n(\omega)| + |\bar{L}_t(\omega)| + |\bar{\mathbf{A}}_t(\omega)|\right) \le M(\omega) < \infty,$$

and from (4.2) in Assumption 4.1 we deduce that for each $\omega \in \mathcal{G}$ there exists a constant $K(\omega)$ such that

$$\sup_n \sup_{t\in[0,T]} \left(\left|\nabla_i h^i(\bar{L}_t^n(\omega), \bar{\mathbf{A}}_t^n(\omega))\right| + \left|\nabla_i h^i(\bar{L}_t(\omega), \bar{\mathbf{A}}_t(\omega))\right|\right) \le K(\omega) < \infty.$$

Hence, for any given $\omega \in \mathcal{G}$, the bounded continuous function $\nabla_i h^i(l, a) \wedge K(\omega)$ coincides with the function $\nabla_i h^i(l, a)$ when evaluated along the sequence $(\bar{L}_s^n(\omega), \bar{\mathbf{A}}_s^n(\omega))$ and at the limit point $(\bar{L}_s(\omega), \bar{\mathbf{A}}_s(\omega))$. Therefore, in what follows we will consider $\omega \in \mathcal{G}$ fixed (we will not stress anymore the dependence on it in the following), and we will assume that $\nabla_i h^i$ is bounded by K.

For $m \in \mathbb{N}$ and $j = 1, ..., m$, consider now the bounded continuous functions $\varphi^{j,n} : [0, T] \to \mathbb{R}$ defined by

$$\varphi_s^{j,n} := \mathbb{1}_{\left(\frac{(j-1)T}{m}, \frac{jT}{m}\right]}(s) \left(s - \frac{(j-1)T}{m}\right) \frac{m}{T} + \mathbb{1}_{\left(\frac{jT}{m}, T\right]}(s).$$

Since each $t \in [0,T]$ belongs to some interval $\left(\frac{(j_t-1)T}{m}, \frac{j_t T}{m}\right]$ for some $j_t \in \{1,...,m\}$, we have

$$
\begin{aligned}
(\text{A.15}) \qquad & \left| \int_t^T \left(\nabla_i h^i(\bar{L}^n_s, \bar{\mathbf{A}}^n_s) - \nabla_i h^i(\bar{L}_s, \bar{\mathbf{A}}_s) \right) ds \right| \\
&= \left| \int_0^T \mathbb{1}_{[t,T]}(s) \left(\nabla_i h^i(\bar{L}^n_s, \bar{\mathbf{A}}^n_s) - \nabla_i h^i(\bar{L}_s, \bar{\mathbf{A}}_s) \right) ds \right| \\
&\leq \left| \int_0^T \left(\mathbb{1}_{[t,T]}(s) - \varphi^{j_t,m}_s \right) \left(\nabla_i h^i(\bar{L}^n_s, \bar{\mathbf{A}}^n_s) - \nabla_i h^i(\bar{L}_s, \bar{\mathbf{A}}_s) \right) ds \right| \\
&\quad + \left| \int_0^T \varphi^{j_t,m}_s \left(\nabla_i h^i(\bar{L}^n_s, \bar{\mathbf{A}}^n_s) - \nabla_i h^i(\bar{L}_s, \bar{\mathbf{A}}_s) \right) ds \right| \\
&\leq \frac{2\,T\,K}{m} + \left| \int_0^T \varphi^{j_t,m}_s \left(\nabla_i h^i(\bar{L}^n_s, \bar{\mathbf{A}}^n_s) - \nabla_i h^i(\bar{L}_s, \bar{\mathbf{A}}_s) \right) ds \right|.
\end{aligned}
$$

Fix now $\varepsilon > 0$ and take $\bar{m} := \bar{m}(\omega)$ large enough such that $2\,T\,K/\bar{m} < \varepsilon/2$.

For each $j = 1, ..., \bar{m}$, the function $\varphi^{j,\bar{m}}_s \nabla_i h^i(l,a)$ is bounded and continuous. Hence, by the convergence established in (4.12), and the characterization (B.2), we can find $\bar{n}(j) := \bar{n}(j,\omega) \in \mathbb{N}$, such that

$$
\left| \int_0^T \varphi^{j,\bar{m}}_s \left(\nabla_i h^i(\bar{L}^n_s, \bar{\mathbf{A}}^n_s) - \nabla_i h^i(\bar{L}_s, \bar{\mathbf{A}}_s) \right) ds \right| < \varepsilon/2, \quad \forall\, n \geq \bar{n}(j).
$$

Since the js are at most $\bar{m}$, we can find $\bar{n} := \bar{n}(\omega)$ large enough such that

$$
\left| \int_0^T \varphi^{j,\bar{m}}_s \left(\nabla_i h^i(\bar{L}^n_s, \bar{\mathbf{A}}^n_s) - \nabla_i h^i(\bar{L}_s, \bar{\mathbf{A}}_s) \right) ds \right| < \varepsilon/2, \quad \forall\, n \geq \bar{n}, \quad \forall\, j \leq \bar{m}.
$$

With this choice of $\bar{n}$, from (A.15) we find that,

$$
\left| \int_t^T \left(\nabla_i h^i(\bar{L}^n_s, \bar{\mathbf{A}}^n_s) - \nabla_i h^i(\bar{L}_s, \bar{\mathbf{A}}_s) \right) ds \right| < \varepsilon,
$$

and, since $\bar{n}$ is independent of t,

$$
\sup_{t \in [0,T]} \left| \int_t^T \left(\nabla_i h^i(\bar{L}^n_s, \bar{\mathbf{A}}^n_s) - \nabla_i h^i(\bar{L}_s, \bar{\mathbf{A}}_s) \right) ds \right| < \varepsilon, \quad \forall\, n \geq \bar{n};
$$

that is, (A.14) is proved.

Now, by continuity of $\nabla_i g^i$ and the convergence at the terminal point T established in (4.13), the sequence $\nabla_i g^i(\bar{L}^n_T, \bar{\mathbf{A}}^n_T)$ converges $\bar{\mathbb{Q}}$-a.s. to $\nabla_i g^i(\bar{L}_T, \bar{\mathbf{A}}_T)$. Finally, by (4.11), we have the uniform convergence of the sequence $\bar{f}^{i,n}$. Hence, we conclude that, for every $i = 1, ..., N$, $\bar{Y}^{i,n}$ converge to $\bar{Y}^i$, uniformly on the interval $[0,T]$, $\bar{\mathbb{Q}}$-a.s. □

Appendix B. Meyer-Zheng Convergence

In this appendix we recall some fact about the so-called Meyer-Zheng topology (see [54]) and we provide some results concerning the tightness of càdlàg processes in such a topology.

Pseudopath topology. For a generic $m \in \mathbb{N}$, let $\mathcal{D}^m[0,\infty)$ be the space of $\mathbb{R}^m$-valued càdlàg functions on $[0,\infty)$, with the Borel σ-algebra generated by the Skorokhod topology. On the half line $[0,\infty)$, consider the measure λ given by $d\lambda := e^{-t}dt$, where dt denotes the Lebesgue measure on $\mathbb{R}$. The *pseudopath topology* τ_{pp} on $\mathcal{D}^m[0,\infty)$ is the topology induced by the convergence in the measure λ on the interval $[0,\infty)$. Notice that we introduce the pseudo-path topology through its characterization proved in Lemma 1 in [54]. Furthermore, recall that we have defined (cf. Subsection 4.3) the pseudopath topology τ_{pp}^T on the space $\mathcal{D}^m$ as the topology induced by the convergence in the measure $dt + \delta_T$ on the interval $[0, T]$, where δ_T denotes the Dirac measure at the terminal point T. Observe that both the topologies τ_{pp} and τ_{pp}^T are metrizable.

Define now the space $\widetilde{\mathcal{D}}^m[0, \infty)$ as the set of elements of $\mathcal{D}^m[0, \infty)$ which are constant on $[T, \infty)$, and notice that $\widetilde{\mathcal{D}}^m[0, \infty)$ is a closed subset of $\mathcal{D}^m[0, \infty)$. Also, observe that the *extension map* $\Psi : \mathcal{D}^m[0, T] \to \widetilde{\mathcal{D}}^m[0, \infty)$, defined by

$$\Psi(x)_t := \begin{cases} x_t & \text{if} \quad t \in [0, T] \\ x_T & \text{if} \quad t \in (T, \infty), \end{cases} \tag{B.1}$$

is an omeomorphism between the topological spaces $(\mathcal{D}^m, \tau_{pp}^T)$ and $(\widetilde{\mathcal{D}}^m[0, \infty), \tau_{pp})$.

In the same way, define the space $\widetilde{\mathcal{D}}^m_\uparrow[0, \infty)$ as the set of elements of $\widetilde{\mathcal{D}}^m[0, \infty)$ which are nondecreasing and nonnegative. Notice that $\widetilde{\mathcal{D}}^m_\uparrow[0, \infty)$ is a closed subset of $\mathcal{D}^m[0, \infty)$ and that the extension map Ψ gives an omeomorphism between the topological spaces $(\mathcal{D}^m_\uparrow, \tau_{pp}^T)$ and $(\widetilde{\mathcal{D}}^m_\uparrow[0, \infty), \tau_{pp})$.

If $\{x^n\}_{n \in \mathbb{N}}$ is a sequence of functions in $\mathcal{D}^m$ converging to a function $x \in \mathcal{D}^m$ in the pseudopath topology τ_{pp}^T, then we have that (see, e.g., Appendix A.3. at p. 116 in [47])

$$\lim_n \int_0^T \phi(s, x_s^n)\, ds = \int_0^T \phi(s, x_s)\, ds, \quad \text{and} \quad \lim_n x_T^n = x_T, \tag{B.2}$$

for each bounded continuous function $\phi : [0, T] \times \mathbb{R}^m \to \mathbb{R}$.

Meyer-Zheng topology and tightness criteria. The *Meyer-Zheng topology* on $\mathcal{P}(\mathcal{D}^m[0, \infty))$ is the topology of weak convergence of probability measures on the topological space $(\mathcal{D}^m[0, \infty), \tau_{pp})$; in the same way we define the Meyer-Zheng topology on $\mathcal{P}(\mathcal{D}^m)$ as the topology of weak convergence of probability measures on the topological space $(\mathcal{D}^m, \tau_{pp}^T)$.

For a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ consider a càdlàg process $X : \Omega \times [0, T] \to \mathbb{R}^m$, and consider the *conditional variation* of X over the interval $[0, T]$, defined as

$$V_T^{\mathbb{P}}(X) := \sup \sum_{i=1}^n \mathbb{E}\left[\left| \mathbb{E}[X_{t_i} - X_{t_{i-1}} | \mathcal{F}_{t_{i-1}}] \right| \right] + \mathbb{E}[|X_{t_n}|], \tag{B.3}$$

where the supremum is taken over all the partitions $0 = t_0 < ... < t_n \leq T$, $n \in \mathbb{N}$. Moreover, for a càdlàg process $X : \Omega \times [0, \infty) \to \mathbb{R}^m$, define the conditional variation of X as

$$V^{\mathbb{P}}(X) := \sup \sum_{i=1}^n \mathbb{E}\left[\left| \mathbb{E}[X_{t_i} - X_{t_{i-1}} | \mathcal{F}_{t_{i-1}}] \right| \right], \tag{B.4}$$

where the supremum is taken over all the partitions $0 = t_0 < ... < t_n = \infty$, $n \in \mathbb{N}$, of $[0, \infty)$, and where have we set $X_\infty = 0$. We recall the following tightness criteria (see [54], Theorem 4).

Theorem B.1 (Meyer and Zheng, 1984)**.** *Let $\{X^n\}_{n \in \mathbb{N}}$ be a sequence of $\mathbb{R}^m$-valued càdlàg processes such that*

$$\sup_n V^{\mathbb{P}}(X^n) < \infty.$$

Then $\{\mathbb{P} \circ X^n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{D}^m[0, \infty))$

We finally prove, for the sake of completeness, a slightly different version of the theorem above that is useful in many occasions during our study.

Lemma B.2. *The following tightness criteria hold true.*

(1) *Let $\{X^n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^m$-valued càdlàg processes defined on $[0,T]$ such that*

$$\sup_n V_T^{\mathbb{P}}(X^n) < \infty.$$

Then $\{\mathbb{P}\circ X^n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{D}^m)$.

(2) *Let $\{X^n\}_{n\in\mathbb{N}}$ be a sequence of nondecreasing, nonnegative, $\mathbb{R}^m$-valued càdlàg processes defined on $[0,T]$ such that*

$$\sup_n \mathbb{E}[|X_T^n|] < \infty.$$

Then $\{\mathbb{P}\circ X^n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{D}^m_\uparrow)$.

Proof. We will prove only the claim (1), the proof of claim (2) follows by an analogous rationale. Using the extension map Ψ defined in (B.1), we have that

$$\sup_n V^{\mathbb{P}}(\Psi(X^n)) = \sup_n V_T^{\mathbb{P}}(X^n) < \infty.$$

Then, we can invoke Theorem B.1 to deduce that the sequence $\{\mathbb{P}\circ\Psi(X^n)\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{D}^m[0,\infty))$. Furthermore, since $\widetilde{\mathcal{D}}^m[0,\infty)$ is a closed subset of $\mathcal{D}^m[0,\infty)$, we have that $\{\mathbb{P}\circ\Psi(X^n)\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\widetilde{\mathcal{D}}^m[0,\infty))$. This means that, for each $\varepsilon>0$, there exists a compact set K^ε in the topological space $(\widetilde{\mathcal{D}}^m[0,\infty),\tau_{pp})$ such that

$$\mathbb{P}[\Psi(X^n)\in K^\varepsilon] \geq 1-\varepsilon \quad \text{for each} \quad n\in\mathbb{N}.$$

Now, since the map Ψ is an omeomorphism, for each $\varepsilon>0$ we have that $\Psi^{-1}(K^\varepsilon)$ is a compact subset of the topological space $(\mathcal{D}^m,\tau_{pp}^T)$, and

$$\mathbb{P}[X^n\in\Psi^{-1}(K^\varepsilon)] = \mathbb{P}[\Psi(X^n)\in K^\varepsilon] \geq 1-\varepsilon \quad \text{for each} \quad n\in\mathbb{N};$$

that is, the sequence $\{\mathbb{P}\circ X^n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{D}^m)$ in the Meyer-Zheng topology. □

We finally summarize in a lemma a result on the convergence of stochastic integrals.

Lemma B.3. *Let $\{F^n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^m$-valued continuous processes which converges $\mathbb{P}$-a.s. to an $\mathbb{R}^m$-valued continuous process F uniformly on $[0,T]$. Let $\{X^n\}_{n\in\mathbb{N}}$ be a sequence of nondecreasing, nonnegative, $\mathbb{R}^m$-valued càdlàg processes defined on $[0,T]$, which converges $\mathbb{P}$-a.s. to nondecreasing, nonnegative, $\mathbb{R}^m$-valued cadlag process X in the pseudopath topology τ_{pp}^T. Suppose, moreover, that there exists two constant $\alpha,p>1$ such that*

$$\sup_n \mathbb{E}\left[\sup_{t\in[0,T]}(|F_t^n|^{\alpha p}+|F_t|^{\alpha p}) + |X_T^n|^{\frac{\alpha p}{p-1}} + |X_T|^{\frac{\alpha p}{p-1}}\right] < \infty. \tag{B.5}$$

Then

$$\lim_n \mathbb{E}\bigg[\int_{[0,T]} F_t^n\, dX_t^n\bigg] = \mathbb{E}\bigg[\int_{[0,T]} F_t\, dX_t\bigg]. \tag{B.6}$$

Proof. We will prove that for each subsequence of indexes there exists a further subsequence for which the limit in (B.6) holds true.

Consider then a subsequence of indexes (not relabeled). We organize the rest of the proof in three steps.

(Step 1) There exists a further subsequence of indexes n_j and a random variable $Z\in\mathbb{L}^\alpha(\bar{\mathbb{Q}})$ such that

$$\lim_j \mathbb{E}\bigg[\int_{[0,T]} F_t^{n_j} dX_t^{n_j}\bigg] = \mathbb{E}[Z]. \tag{B.7}$$

Notice that there exists a suitable constant $\tilde{C} > 0$ such that

$$\begin{aligned}\text{(B.8)} \qquad \mathbb{E}\bigg[\bigg|\int_{[0,T]} F_t^n dX_t^n\bigg|^\alpha\bigg] &\leq \tilde{C}\mathbb{E}\bigg[\sup_{t\in[0,T]} |F_t^n|^\alpha |X_T^n|^\alpha\bigg] \\ &\leq \tilde{C}\bigg(\mathbb{E}\bigg[\sup_{t\in[0,T]} |F_t^n|^{\alpha p}\bigg]\bigg)^{\frac{1}{p}} \Big(\mathbb{E}\Big[|X_T^n|^{\frac{\alpha p}{p-1}}\Big]\Big)^{\frac{p-1}{p}}.\end{aligned}$$

Using now (B.5) we find

$$\sup_n \mathbb{E}\bigg[\bigg|\int_{[0,T]} F_t^n dX_t^n\bigg|^\alpha\bigg] < \infty; \tag{B.9}$$

that is, that the sequence $\{\int_{[0,T]} F_t^n dX_t^n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^\alpha(\bar{\mathbb{Q}})$.

Since $\alpha > 1$, by the reflexivity of $\mathbb{L}^\alpha(\bar{\mathbb{Q}})$, there exists a subsequence $\{\int_{[0,T]} F_t^{n_j} dX_t^{n_j}\}_{j\in\mathbb{N}}$ and a random variable $Z \in \mathbb{L}^\alpha(\bar{\mathbb{Q}})$, for which the limit in (B.7) holds true.

(Step 2) We have that

$$\lim_j \mathbb{E}\bigg[\int_{[0,T]} F_t\, dX_t^{n_j}\bigg] = \mathbb{E}[Z]. \tag{B.10}$$

Following a rationale similar to that yielding (B.9), it is possible to prove the following uniform integrability estimates

$$\sup_j \mathbb{E}\bigg[\bigg|\int_{[0,T]} F_t\, dX_t^{n_j}\bigg|^\alpha\bigg] < \infty. \tag{B.11}$$

Furthermore, by assumption, the sequence $\{|X_T^n|\}_{n\in\mathbb{N}}$ is $\mathbb{P}$-a.s. convergent, hence bounded, so that

$$\lim_j \sup_{t\in[0,T]} |F_t - F_t^{n_j}|\,|X_T^{n_j}| = 0, \quad \mathbb{P}\text{-a.s.} \tag{B.12}$$

Now, thanks to the limit in (B.12) and to the uniform integrability estimates (B.11) and (B.9), we deduce that

$$\lim_j \mathbb{E}\bigg[\int_{[0,T]} (F_t - F_t^{n_j})\, dX_t^{n_j}\bigg] = 0,$$

and from (B.7), we conclude that

(B.13)

$$\lim_j \mathbb{E}\bigg[\int_{[0,T]} F_t\, dX_t^{n_j}\bigg] = \lim_j \mathbb{E}\bigg[\int_{[0,T]} (F_t - F_t^{n_j})\, dX_t^{n_j}\bigg] + \lim_j \mathbb{E}\bigg[\int_{[0,T]} F_t^{n_j}\, dX_t^{n_j}\bigg] = \mathbb{E}[Z],$$

which completes the proof of *Step 2*.

(Step 3) $\mathbb{E}[Z]$ *coincides with the right-hand side of (B.6).*

Fix $\delta > 0$. We extend by zero the processes X^n and X on the interval $[-\delta, 0)$; that is, for each $n \in \mathbb{N}$ and $t \in [-\delta, 0)$, we set $X_t^n := 0$ and $X_t := 0$. Furthermore, we extend by continuity the processes F^n and F on the interval $[-\delta, 0)$; that is, for each $n \in \mathbb{N}$ and $t \in [-\delta, 0)$, we set $F_t^n := F_0^n$ and $F_t := F_0$. Thanks to our assumption, we have that $\bar{\mathbb{Q}}$-a.s.

$$X^n \to X \quad \text{in the measure} \quad dt + \delta_T \quad \text{on the interval} \quad [-\delta, T]. \tag{B.14}$$

Moreover, since by Condition (B.5) the sequence $\{X_T^{n_j}\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{L}^1(\bar{\mathbb{Q}})$, then, by (a minimal adjustment to $[-\delta, T]$ of) Lemma 3.5 in [39], there exist a nondecreasing, nonnegative,

$\mathbb{R}^m$-valued càdlàg process B defined on $[-\delta, T]$ and a subsequence (not relabeled) of $\{X^{n_j}\}_{j\in\mathbb{N}}$ such that, $\bar{\mathbb{Q}}$-a.s.,

$$\lim_m \int_{-\delta}^T \varphi_t dB_t^m = \int_{-\delta}^T \varphi_t dB_t \quad \forall\, \varphi \in \mathcal{C}_b((-\delta, T); \mathbb{R}^d) \quad \text{and} \quad \lim_m B_T^m = B_T, \tag{B.15}$$

where we have set, $\bar{\mathbb{Q}}$-a.s.

$$B_t^m := \frac{1}{m}\sum_{j=1}^m X_t^{n_j}, \qquad \forall t \in [-\delta, T]. \tag{B.16}$$

Now, thanks to the convergence in (B.14) and to (an analogous for the interval $[-\delta, T]$ of) (B.2), we deduce that $\bar{\mathbb{Q}}$-a.s.

$$\lim_j \int_{-\delta}^T \phi(s, X_s^{n_j})\, ds = \int_{-\delta}^T \phi(s, X_s)\, ds,$$

for each bounded continuous function $\phi : [-\delta, T]\times\mathbb{R}^d \to \mathbb{R}$. Hence, letting $\varphi \in \mathcal{C}_c^\infty((-\delta, T); \mathbb{R}^d)$, and recalling that the sequence $\{|X_T^n|\}_{n\in\mathbb{N}}$ is $\mathbb{P}$-a.s. bounded, an integration by parts reveals that, $\bar{\mathbb{Q}}$-a.s.,

$$\int_{-\delta}^T \varphi_t dX_t = -\int_{-\delta}^T X_t \varphi_t' dt = -\lim_j \int_{-\delta}^T X_t^{n_j} \varphi_t' dt,$$

The latter, together with (B.15), implies that (again using integration by parts), $\bar{\mathbb{Q}}$-a.s.,

$$\int_{-\delta}^T \varphi_t dB_t = \lim_m \frac{1}{m}\sum_{j=1}^m \int_{-\delta}^T \varphi_t dX_t^{n_j} = -\lim_m \frac{1}{m}\sum_{j=1}^m \int_{-\delta}^T X_t^{n_j}\varphi_t' dt = \int_{-\delta}^T \varphi_t dX_t.$$

Therefore, by the fundamental lemma of the Calculus of Variation (see Theorem 1.24 at p. 26 in [25]) and by right-continuity of X and B we deduce that $B_t = X_t$ for all $t \in [-\delta, T)$. Finally, using the convergence at the terminal point T in the second equation of (B.15), we deduce that

$$B_t = X_t, \qquad \forall t \in [-\delta, T], \qquad \bar{\mathbb{Q}}\text{-a.s.} \tag{B.17}$$

Next, by (B.11), since $\alpha > 1$, we find that

$$\sup_m \mathbb{E}\left[\left|\int_{-\delta}^T F_t\, dB_t^m\right|^\alpha\right] \le \sup_m \frac{1}{m}\sum_{j=1}^m \mathbb{E}\left[\left|\int_{-\delta}^T F_t\, dX_t^{n_j}\right|^\alpha\right] \le \sup_j \mathbb{E}\left[\left|\int_{-\delta}^T F_t\, dX_t^{n_j}\right|^\alpha\right] < \infty, \tag{B.18}$$

which implies that the sequence $\{\int_{-\delta}^T F_t d\bar{B}_t^m\}_{m\in\mathbb{N}}$ is uniformly integrable. Moreover, by (B.5), it follows that, $\mathbb{P}$-a.s., F is a bounded (by a constant depending on ω) and continuous function on the interval $[-\delta, T]$. Hence by (B.15) and (B.17) we have, $\bar{\mathbb{Q}}$-a.s.,

$$\lim_m \int_{-\delta}^T F_t\, dB_t^m = \int_{-\delta}^T F_t\, dX_t. \tag{B.19}$$

Furthermore, from the pointwise limit (B.19) and the uniform integrability estimate (B.18), we have

$$\lim_m \mathbb{E}\left[\int_{-\delta}^T F_t\, dB_t^m\right] = \mathbb{E}\left[\int_{-\delta}^T F_t\, dX_t\right].$$

This allows to deduce, using (B.10) of *Step 2* and the fact that the processes X^{n_j} are constantly null on $[-\delta, 0)$, that

$$\mathbb{E}[Z] = \lim_m \frac{1}{m} \sum_{j=1}^{m} \mathbb{E}\left[\int_{-\delta}^{T} F_t \, dX_t^{n_j}\right] = \lim_m \mathbb{E}\left[\int_{-\delta}^{T} F_t \, dB_t^m\right] = \mathbb{E}\left[\int_{-\delta}^{T} F_t \, dX_t\right].$$

Since now the process X_t is null on $[-\delta, 0)$, we conclude from the latter that

$$\mathbb{E}\left[\int_{[0,T]} F_t \, dX_t\right] = \mathbb{E}[Z],$$

which is in fact the thesis of *Step 3*. □

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